

# Engineering Bernoulli Equation

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The Engineering Bernoulli equation can be derived from the principle of conservation of energy. Several books provide such a derivation in detail. The interested student is encouraged to consult White (1) or Denn (2). Here, I have merely summarized the important forms of this equation for your use in solving problems. Whenever you use this equation, be sure to draw a sketch and clearly mark the datum from which heights are measured. When setting a term to zero, indicate the reason for doing so. For example, when the free surface of the liquid in a tank is exposed to the atmosphere, or when it is issuing as a free jet into the atmosphere, the pressure at that location is set equal to zero gage. When liquid is taken out of a vessel through a pipe of cross-sectional area that is small compared with that of the vessel, the velocity of the free surface will be relatively small, and the kinetic energy term associated with that velocity can be set equal to zero without much error.

When the Engineering Bernoulli Equation is applied to fluid contained in a control volume fixed in space, typically the control volume has impenetrable boundaries, with the exception of one or more inlets and one or more outlets through which fluid enters and leaves the control volume. During passage of fluid through the control volume, mechanical work is irreversibly transformed by fluid friction into heat, leading to “losses.” Also, the fluid may operate a turbine, performing work on the blades of the machine, or work may be performed on the fluid by a pump. These lead to “shaft work,” assumed by convention to be positive when performed by the fluid, and negative when performed on the fluid. Both losses and shaft work are included in the energy form of the Engineering Bernoulli Equation on the basis of unit mass of fluid flowing through.

The two most common forms of the resulting equation, assuming a single inlet and a single exit, are presented next.

## Energy Form

Here is the “energy” form of the Engineering Bernoulli Equation. Each term has dimensions of energy per unit mass of fluid.

$$\frac{p_{out}}{\rho} + \frac{V_{out}^2}{2} + gz_{out} = \frac{p_{in}}{\rho} + \frac{V_{in}^2}{2} + gz_{in} - \text{loss} - w_s$$

In the above equation,  $p$  is pressure, which can be either absolute or gage, but should be in the same basis on both sides,  $\rho$  represents the density of the fluid, assumed constant,  $V$  is the velocity of the fluid at the inlet/outlet, and  $z$  is the elevation about a datum that is specified. Note that it is only differences in elevation that matter, so that the choice of the datum for  $z$  is arbitrary. The symbol  $g$  stands for the magnitude of the acceleration due to gravity.

The term “loss” stands for losses per unit mass flowing through, while  $w_s$  represents the shaft work done **by** the fluid per unit mass flowing through. The form given above assumes flat velocity profiles across the inlet and exit, which is a reasonable approximation in turbulent flow. In laminar flow, the velocity distribution across the cross-section must be accommodated in the kinetic energy calculation. In that case, we use the average velocities at the inlet and exit, but multiply the kinetic energy terms on each side of the Engineering Bernoulli Equation by a correction factor  $\alpha$  that accounts for the variation of the kinetic energy of the fluid across the cross-section. You can consult references (1) or (2) to learn how to calculate this correction factor.

We express loss usually as a certain number ( $N$ ) of velocity heads. In this case, loss =  $N(V^2 / 2)$ . At other times, loss is expressed as a certain ( $M$ ) head of fluid. In this case, loss =  $gM$ .

## Head Form

The “head” form of the Engineering Bernoulli Equation is obtained by dividing the energy form throughout by the magnitude of the acceleration due to gravity,  $g$ .

$$\frac{P_{out}}{\gamma} + \frac{V_{out}^2}{2g} + z_{out} = \frac{P_{in}}{\gamma} + \frac{V_{in}^2}{2g} + z_{in} - \frac{\text{loss}}{g} - \frac{w_s}{g}$$

In this equation, the symbol  $\gamma$  represents the specific weight of fluid.  $\gamma = \rho g$

We define the head developed by a pump as  $h_p = -w_s / g$ . Because the work term  $w_s$  is negative for a pump, the head developed by a pump  $h_p$  is always positive.

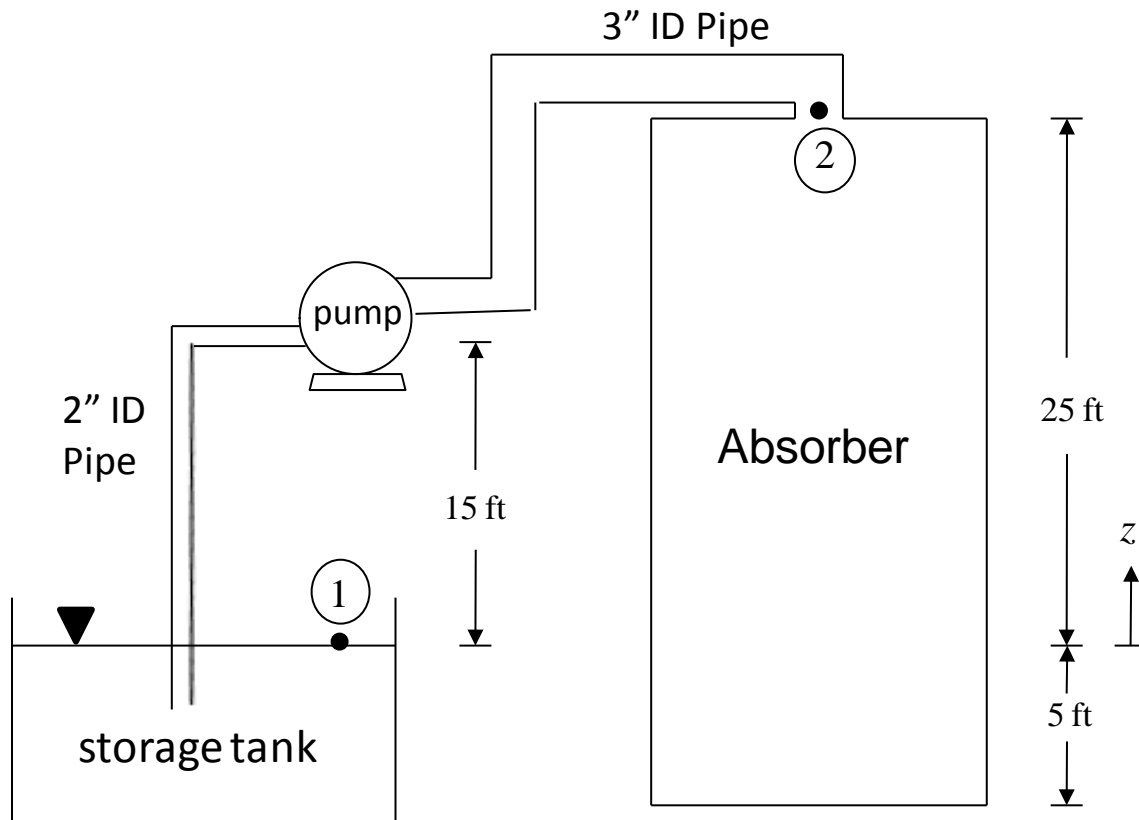
Loss is always positive. We define a head loss term as  $\frac{\text{loss}}{g} = h_{\text{friction}} = h_f$ .

Therefore, we can rewrite the head form of the Engineering Bernoulli Equation as

$$\frac{P_{out}}{\gamma} + \frac{V_{out}^2}{2g} + z_{out} = \frac{P_{in}}{\gamma} + \frac{V_{in}^2}{2g} + z_{in} - h_f + h_p$$

Now, two examples are presented that will help you learn how to use the Engineering Bernoulli Equation in solving problems. In a third example, another use of the Engineering Bernoulli equation is illustrated.

## Example 1: Sizing a Pump



The pump shown here is used to lift a process liquid of density  $1.93 \text{ slug} / \text{ft}^3$  from a storage tank, and discharge it at a rate of  $0.75$  cubic feet per second into the top of an absorber. The inlet to the absorber is located  $25$  feet above the free surface of the liquid in the storage tank, and the pump inlet is located at an elevation of  $15$  feet above that of the free surface. You can assume that the absorber operates at atmospheric pressure.

A  $2$ " ID pipe leads from the storage tank to the pump, while the pipe from the pump to the top of the absorber is of ID  $3$ ". You can assume the losses in the  $2$ " ID pipe to be  $4$  velocity heads, and the losses in the  $3$ " ID pipe to be  $5$  velocity heads. Assuming the pump is  $85\%$  efficient, calculate the BHP (Brake Horse Power) of the pump.

## Solution

First, we must identify locations 1 and 2 for applying the Engineering Bernoulli Equation. Recall that the pressure, velocity, and elevation at each of these locations appear in the Engineering Bernoulli Equation. Thus, we select these locations in such a way as to be able to specify the maximum amount of information possible at each. With this in mind, location 1 is selected to be at the free surface of the liquid in the storage tank, and location 2 at the entrance to the absorber. Further, we select the datum for measuring heights to be at the free surface of the liquid in the storage tank, as shown in the sketch. Note that the height of the pump above that free surface is given, but it is not a good idea to choose our location 2 at either the inlet or the exit of the pump, because it would unnecessarily add to the calculational burden.

Now list all the known information at the two locations.

$$p_1 = 0 \text{ gage (Open to atmosphere)}$$

$$V_1 = 0 \text{ (Large cross-sectional area)}$$

$$z_1 = 0 \text{ (By choice of datum)}$$

From the given discharge rate and the diameter of the pipe at the absorber inlet, we can calculate

$$V_2. \quad Q = 0.75 \frac{\text{ft}^3}{\text{s}} = V_2 A_2 = V_2 \times \frac{\pi}{4} (0.25 \text{ ft})^2 \text{ yielding } \boxed{V_2 = 15.3 \text{ ft/s}}.$$

$$p_2 = 0 \text{ gage (Absorber is at atmospheric pressure)}$$

$$V_2 = 15.3 \text{ ft/s (from specified data)}$$

$$z_2 = 25 \text{ ft (specified)}$$

Let us write the Engineering Bernoulli Equation. We use location 1 for “in” and location 2 for “out.”

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 - \text{loss} - w_s$$

Substituting some of the known information into the above equation, we obtain

$$0 + \frac{V_2^2}{2} + gz_2 = 0 + 0 + gz_1 - \text{loss} - w_s$$

$$\text{or } \boxed{-w_s = \text{loss} + \frac{V_2^2}{2} + g(z_2 - z_1)}$$

The implication of the above equation is as follows. The shaft work delivered by the pump to the fluid accounts for the losses in the flow through the pipes and any fittings, the exit velocity

head that must be delivered, and the lifting of the fluid from elevation  $z_1$  to elevation  $z_2$ . If the fluid enters the control volume with some kinetic energy (that is, if  $V_1 \neq 0$ ), then that kinetic energy would help reduce the shaft work needed. Likewise, we can see from the Engineering Bernoulli Equation that the shaft work also must supply any needed pressurizing of the fluid  $(p_2 - p_1) / \rho$ . In this problem,  $p_1 = p_2$ . We have chosen each to be zero by using gage pressures, but even if we had used absolute pressures, the difference would still be zero and no shaft work would be needed for increasing the pressure.

Let us first calculate the loss. It is specified as 4 velocity heads in the 2" ID pipe, and 5 velocity heads in the 3" ID pipe. We already calculated the velocity in the 3" ID pipe to be  $V_2 = 15.3 \text{ ft/s}$ . Because  $Q = VA$ , and the cross sectional areas are proportional to the square of the diameters, we can write

$$V_{2" \text{ ID pipe}} = V_{3" \text{ ID pipe}} \left( \frac{3 \text{ inches}}{2 \text{ inches}} \right)^2 = 15.3 \frac{\text{ft}}{\text{s}} \times \frac{9}{4} = \boxed{34.4 \frac{\text{ft}}{\text{s}}}$$

Therefore, we can find the loss as

$$\text{loss} = 4 \frac{V_{2" \text{ ID pipe}}^2}{2} + 5 \frac{V_{3" \text{ ID pipe}}^2}{2} = 4 \times \frac{\left( 34.4 \frac{\text{ft}}{\text{s}} \right)^2}{2} + 5 \times \frac{\left( 15.3 \frac{\text{ft}}{\text{s}} \right)^2}{2} = \boxed{2.95 \times 10^3 \frac{\text{ft}^2}{\text{s}^2}}$$

Substituting in the result for the shaft work,

$$\begin{aligned} -w_s &= \text{loss} + \frac{V_2^2}{2} + g(z_2 - z_1) = 2.95 \times 10^3 \left( \frac{\text{ft}^2}{\text{s}^2} \right) + \frac{\left( 15.3 \frac{\text{ft}}{\text{s}} \right)^2}{2} + 32.2 \left( \frac{\text{ft}}{\text{s}^2} \right) \times (25 \text{ ft}) \\ &= \boxed{3.87 \times 10^3 \frac{\text{ft}^2}{\text{s}^2}} \end{aligned}$$

The units of the shaft work appear to be strange, but they are not. Let us investigate this further. Recall that each term in this version of the Engineering Bernoulli Equation must have the same units as the loss or shaft work, which are in energy per unit mass flowing through the control volume. Let us work out the units of energy per unit mass in the British system. Energy has the same units as work, which is force times distance. Therefore, the units of shaft work are  $\frac{\text{ft} \cdot \text{lb}_f}{\text{slug}}$

. One  $\text{lb}_f$  is the force required to accelerate a mass of one  $\text{slug}$  by one  $\frac{\text{ft}}{\text{s}^2}$ , i.e.,

$$\boxed{1 \text{ lb}_f = 1 \frac{\text{slug} \cdot \text{ft}}{\text{s}^2}}. \text{ Therefore, the units of shaft work are } \frac{\text{ft} \cdot \text{lb}_f}{\text{slug}} = \frac{\text{ft} \cdot \text{slug} \cdot \text{ft}}{\text{slug} \cdot \text{s}^2} = \frac{\text{ft}^2}{\text{s}^2}. \text{ So,}$$

we see that the units of  $\frac{ft \bullet lb_f}{slug}$  are equivalent to  $\frac{ft^2}{s^2}$ . In the metric system, the corresponding units would be  $\frac{J}{kg}$  and  $\frac{m^2}{s^2}$ , which you should verify are equivalent to each other.

Therefore, the work to be delivered by the pump to the fluid is  $3.87 \times 10^3 \frac{ft \bullet lb_f}{slug}$ . We must now multiply this by the mass flow rate, which is given by

$$\dot{m} = \rho Q = 1.93 \left( \frac{slug}{ft^3} \right) \times 0.75 \left( \frac{ft^3}{s} \right) = \boxed{1.45 \frac{slug}{s}}$$

The power supplied to the fluid by the pump is given by

$$\text{Power to Fluid} = \dot{m} \times (-w_s) = 1.45 \left( \frac{slug}{s} \right) \times 3.87 \times 10^3 \left( \frac{ft \bullet lb_f}{slug} \right) = \boxed{5.61 \times 10^3 \frac{ft \bullet lb_f}{s}}$$

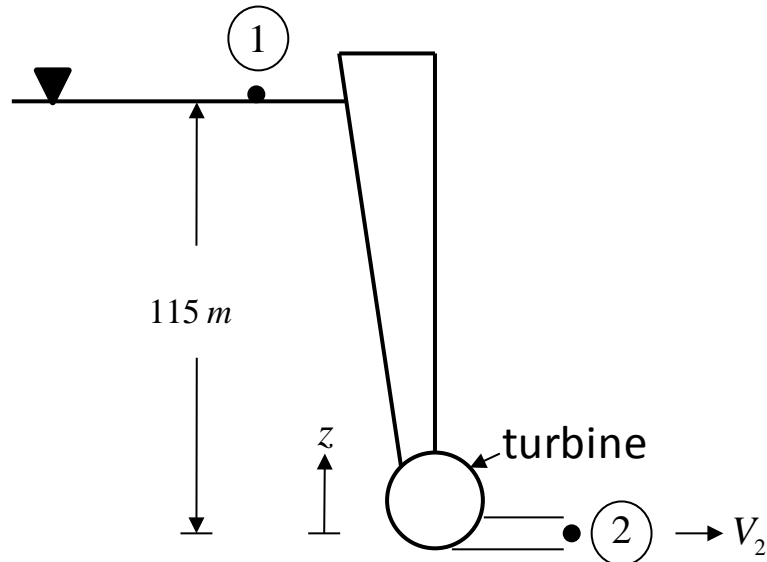
Power is usually reported in Horse Power abbreviated as HP or hp. One Horse Power is equivalent to  $550 \frac{ft \bullet lb_f}{s}$  or 746 W in the metric system. Therefore,

$$\text{Power to Fluid} = 5.61 \times 10^3 \left( \frac{ft \bullet lb_f}{s} \right) \times \frac{1 hp}{550 \left( \frac{ft \bullet lb_f}{s} \right)} = \boxed{10.2 hp}$$

There are losses that occur inside a pump. Therefore, the power supplied to a pump, called the Brake Horse Power or BHP is larger than the power delivered to the fluid. The ratio of the power delivered to the fluid to the Brake Horse Power is termed the efficiency of the pump represented by the symbol  $\eta$ . Therefore,

$$BHP = \frac{\text{Power to Fluid}}{\eta} = \frac{10.2 hp}{0.85} = \boxed{12.4 hp}$$

## Example 2: A Turbine Problem



In the above example problem, a hydroelectric turbine at the base of a dam is shown schematically. The height of the water above the turbine station is given as  $115\text{ m}$ . This turbine produces  $4.6\text{ MW}$  of electricity, and you may assume the losses in the system to be equivalent to  $10\text{ m}$  of head of water. The diameter of the pipe at the turbine exit is  $0.75\text{ m}$ , and the velocity of water exiting from the pipe is  $12.5\text{ m/s}$ . Calculate the efficiency of the turbine.

### Solution

As in the earlier example, we first identify locations 1 and 2 for applying the Engineering Bernoulli Equation. Because the pressure, velocity, and elevation at each of these locations appear in the Engineering Bernoulli Equation, we again select these locations in such a way as to be able to specify the maximum amount of information possible at them. With this in mind, we choose location 1 at the free surface of the liquid in the reservoir, and location 2 at the exit from the turbine. The datum for measuring heights is chosen logically at the base of the reservoir where the turbine is located, as shown in the sketch.

We proceed to list all the known information at the two locations.

$$p_1 = 0 \text{ gage (Open to atmosphere)}$$

$$V_1 = 0 \text{ (Large cross-sectional area)}$$

$$z_1 = 115\text{ m (Specified)}$$

$p_2 = 0$  gage (Free jet into the atmosphere)

$V_2 = 12.5 \text{ m/s}$  (given)

$z_2 = 0$  (By choice of datum)

Using location 1 for “in” and location 2 for “out,” the Engineering Bernoulli Equation is written as follows.

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 - \text{loss} - w_s$$

As in Example 1, we substitute some of the known information into the above equation, yielding

$$\boxed{w_s = g(z_1 - z_2) - \text{loss} - \frac{V_2^2}{2}}$$

This equation tells us that the shaft work we can obtain from the turbine is proportional to the head of water available minus the loss in the system minus the velocity head in the exit stream. If the inlet stream had a significant amount of velocity head, that would help increase the shaft work, but because of the large cross-sectional area of the reservoir, the velocity at the free surface can be assumed to be negligible. We are given the loss as  $10 \text{ m}$  of head of water, so it is easiest to cast it as  $\text{loss} = g \times 10 (m)$ . Substituting the values of  $g$ ,  $z_1$ ,  $z_2$ ,  $V_2$ , and the result for the loss in the equation for the shaft work, we obtain

$$w_s = 9.81 \frac{m}{s^2} \times (115 - 0)(m) - 9.81 \frac{m}{s^2} \times 10(m) - \frac{[12.5(m/s)]^2}{2} = \boxed{952 \frac{m^2}{s^2}}$$

This is the work performed on the turbine by the water per unit mass flowing through. Therefore, we must multiply this by the mass flow rate  $\dot{m}$  to obtain the power delivered by the water to the turbine blades, which can be written as  $\dot{m} = \rho Q = \rho V_2 A_2$ .

To estimate the density of the water, we need the temperature. The temperature is not specified, but we can assume it to lie in the ordinary range of temperatures for the environment. Given the variations of the temperature depending on the time of the year, it is safe to use a density of water  $\rho = 10^3 \text{ kg/m}^3$ , recognizing that this value is subject to an error of a couple of  $\text{kg/m}^3$ . The diameter of the pipe at the turbine exit is given as  $D_2 = 0.75 \text{ m}$ , so that its cross-sectional area is

$$A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} (0.75 \text{ m})^2 = \boxed{0.442 \text{ m}^2}$$



Now, we can proceed to calculate the mass flow rate of water out of the turbine as

$$\dot{m} = \rho V_2 A_2 = 10^3 \left( \frac{\text{kg}}{\text{m}^3} \right) \times 12.5 \left( \frac{\text{m}}{\text{s}} \right) \times 0.442 (\text{m}^2) = \boxed{5.52 \times 10^3 \frac{\text{kg}}{\text{s}}}$$

Therefore, the power delivered to the turbine blades is given by

$$\text{Power to turbine blades} = w_s \times \dot{m} = 952 \left( \frac{\text{m}^2}{\text{s}^2} \right) \times 5.52 \times 10^3 \left( \frac{\text{kg}}{\text{s}} \right) = \boxed{5.26 \times 10^6 \frac{\text{kg} \cdot \text{m}^2}{\text{s}^3}}$$

Let us work out the units of this power.

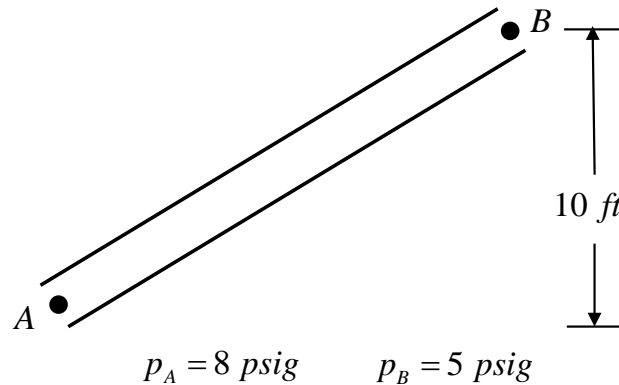
$$1 \frac{\text{kg} \cdot \text{m}^2}{\text{s}^3} = 1 \left( \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \right) \times \left( \frac{\text{m}}{\text{s}} \right) = 1 \left( \frac{\text{N} \cdot \text{m}}{\text{s}} \right) = 1 \left( \frac{\text{J}}{\text{s}} \right) = \boxed{1 \text{ W}}$$

Thus, the power is in Watts. The power output of the turbine is given as  $4.6 \text{ MW}$  or  $4.6 \times 10^6 \text{ W}$ . Therefore, the efficiency of the turbine can be calculated as follows.

$$\text{Efficiency } \eta = \frac{\text{Power output from the turbine}}{\text{Power to turbine blades}} = \frac{4.60 \times 10^6 \text{ W}}{5.26 \times 10^6 \text{ W}} = \boxed{0.875}$$

The efficiency is also reported commonly as a per cent. In that case, we would state it as 87.5%.

### Example 3: Establishing the Flow Direction in a Pipe



What is the direction of flow of water in this pipe? You can assume that the flow is steady, and that the density of water is constant.

#### Solution

We can determine the direction of flow of water in this pipe in a couple of ways. One way is to determine the hydrostatic pressure difference between A and B. If the water is stationary in the pipe, the pressure at A would be larger than that at B by  $\gamma h$ , where  $\gamma$  is the specific weight of the fluid and  $h$  is the elevation difference between A and B. The specific weight of water is

$\gamma_{\text{water}} = 62.5 \frac{\text{lb}_f}{\text{ft}^3}$ . Therefore, we can write

$$(p_A - p_B)_{\text{hydrostatic}} = \gamma_{\text{water}} h = 62.5 \left( \frac{\text{lb}_f}{\text{ft}^3} \right) \times 10 (\text{ft}) = 625 \frac{\text{lb}_f}{\text{ft}^2} = \boxed{4.34 \text{ psi}}$$

But the actual pressure difference  $p_A - p_B = 8 \text{ psig} - 5 \text{ psig} = \boxed{3 \text{ psi}}$  (Note that pressure differences are absolute – they are not measured relative to any other pressure)

So, we see that the pressure at A is not large enough for a hydrostatic balance, and flow must occur from point B to point A.

The second way uses the idea that losses in a flow must always be positive, because they arise from irreversible conversion of mechanical work into heat. To use this idea, we apply the Engineering Bernoulli equation to the system. Let us begin by assuming that the flow occurs from A to B. Therefore, location A is the “in” location, and location B is the “out” location for the control volume and we can write

$$\frac{p_B}{\rho} + \frac{V_B^2}{2} + gz_B = \frac{p_A}{\rho} + \frac{V_A^2}{2} + gz_A - \text{loss} - \cancel{w_s}$$

where we have crossed out the shaft work term, because there is none in the control volume. Because the diameter of the pipe is constant and the density of water can be assumed to be constant,  $V_A = V_B$ . Therefore, we can write

$$\frac{\text{loss}}{g} = \frac{p_A - p_B}{\gamma} - (z_B - z_A) = \frac{(8 - 5) \left( \frac{\text{lb}_f}{\text{in}^2} \right) \times \left( \frac{144 \text{ in}^2}{\text{ft}^2} \right)}{62.5 \frac{\text{lb}_f}{\text{ft}^3}} - 10 \text{ ft} = (6.91 - 10) \text{ ft} = \boxed{-3.09 \text{ ft}}$$

so that the loss is  $-126 \text{ ft}^2 / \text{s}^2$  and is negative. This is not possible. Therefore, our assumption that flow occurs from A to B must be wrong, and the actual flow direction is from B to A. We can confirm the correctness of this conclusion by recasting the flow as occurring from B to A in the Engineering Bernoulli equation. In that case, the result for the loss is

$$\frac{\text{loss}}{g} = (z_B - z_A) - \frac{p_A - p_B}{\gamma} = (10 - 6.91) \text{ ft} = \boxed{3.09 \text{ ft}}$$

Therefore, the loss is positive, and equal to  $+126 \text{ ft}^2 / \text{s}^2$ .

Even though the two approaches we used appear to be different superficially, they are basically the same. To see this, examine the loss calculation when the flow is assumed to be from A to B. The first term on the right side is the amount by which the pressure at A is larger than the pressure at B, expressed in head of water. This is only 6.91 feet, whereas the second term is the actual elevation difference between B and A, which is 10 feet. Thus, the pressure difference is smaller than what we'd see if the fluid were stationary. The loss works out to be the pressure difference between A and B expressed as head of water minus the elevation difference between B and A. This is a negative quantity, and it is impossible to have such a flow.

If we assume the flow direction to be from B to A, which is the correct direction, then the pressure difference still is equivalent to 6.91 feet of water, but the loss now is the elevation difference between B and A minus the pressure difference between A and B in head of water, and becomes positive, telling us that this is the correct direction of flow.

## Bernoulli Equation

In some applications, there is no machinery such as a pump or turbine in the control volume. In such cases, the shaft work term in the Engineering Bernoulli Equation is zero. Furthermore, it is convenient to neglect losses in short sections of piping. In other cases, where there are losses they are accommodated separately using a so-called loss coefficient. To handle these types of applications, we use a simpler version of the Bernoulli Equation that contains neither shaft work nor a loss term. It is simply called the Bernoulli Equation.

### Bernoulli Equation

The energy form is written as follows. Each term has dimensions of energy per unit mass of fluid. We use the convention that location 1 is the inlet and location 2 is the outlet.

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$

The corresponding “Head” form is

$$\frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 = \frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1$$

While it is true that the Bernoulli Equation can be derived rigorously along a streamline for a special class of flows called “potential flow,” this is not particularly important when we apply it to practical problems. Some users try to draw contrived streamlines for this purpose, which are unnecessary. The main idea is that shaft work is zero, and losses in the control volume are treated as being negligible in the actual situation.

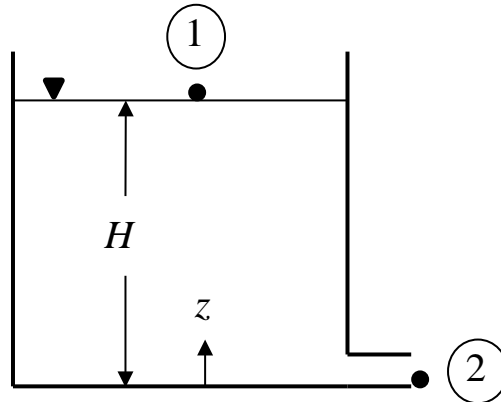
A few situations where the Bernoulli Equation proves useful are listed below.

1. Estimating the flow rate out of a tank, given the diameter of the exit pipe
2. Evaluating flow rates and pressures in a siphon
3. Evaluating the volumetric flow rate through a pipe using a flow measurement device such as a venturimeter
4. Determining a local velocity using the Pitot-Static Tube

Each of these examples is illustrated next.

## Example 1: Tank Draining Through a Pipe

Obtain an expression for the velocity and the volumetric and mass flow rates out of a pipe of diameter  $d$  at the bottom of a storage tank in which a liquid is maintained at a height  $H$ . The tank is open to the atmosphere, and the liquid issues out of the pipe as a free jet into the atmosphere.



As shown in the sketch, we place location 1 at the free liquid surface, and location 2 at the exit of the pipe. Clearly, there is no pump or turbine here. If we neglect losses, which is reasonable if the pipe is short and the storage tank is large and wide, we can use the Bernoulli Equation, written between locations 1 and 2 in the sketch.

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$

The pressures at locations 1 and 2 are both zero gage. By choice of datum,  $z_1 = H$  and  $z_2 = 0$ . Because of the large cross-sectional area at location 1, the velocity  $V_1$  can be set equal to zero. Thus, we can make these substitutions into the Bernoulli Equation to obtain

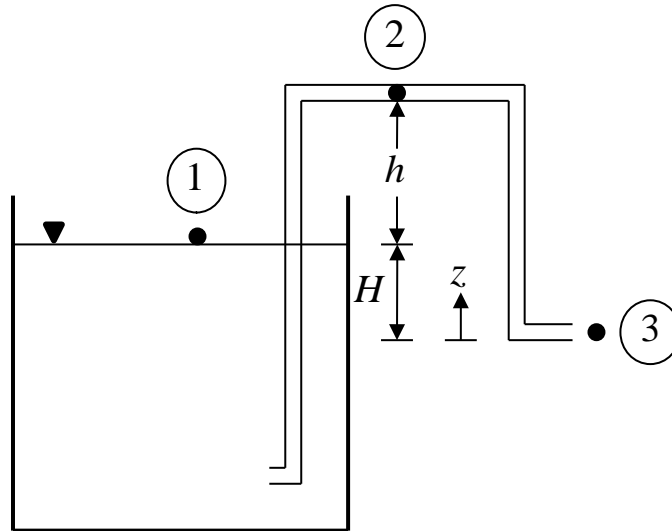
$$0 + \frac{V_2^2}{2} + 0 = 0 + 0 + gH$$

which yields  $V_2 = \sqrt{2gH}$ . Usually, any small losses are accommodated by multiplying this result by a “loss coefficient.” The volumetric flow rate  $Q = (\pi d^2 / 4)V_2$ , and the mass flow rate is given as  $\dot{m} = \rho Q = (\pi d^2 / 4)\rho V_2$ .

Note that we assumed that the liquid level in the tank is maintained constant at  $H$ . The result we obtained can be used even when that level is changing because of loss of liquid through the exit, so long as the rate of decrease of  $H$  is sufficiently small. In fact, the rate of loss of liquid can be used to calculate the rate at which  $H$  decreases with time, using an unsteady mass balance on the contents of the tank. This type of assumption is known as the quasi-steady-state assumption. In other words, we assume “steady” conditions for calculating the exit velocity out of the pipe, even though the conditions are truly unsteady. The quasi-steady assumption is commonly used in heat and mass transfer operations in situations where its use is justified.

## Example 2: Siphon

A siphon is a device for removing liquid from a container using a pipe that rises above the liquid level in the container. A sketch of a typical siphon is shown below.



As we can see there are no pumps or turbines here, and if we neglect the small losses involved, we can apply the Bernoulli Equation to obtain an estimate of the velocity out of the pipe,  $V_3$ . All that matters is the height  $H$  of the free surface of the liquid in the container above the location of the siphon outlet. We have therefore selected the location 3 as datum, and location 1 as the free surface of the liquid in the container. The Bernoulli Equation is

$$\frac{p_3}{\rho} + \frac{V_3^2}{2} + gz_3 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$

Because locations 1 and 3 are both open to the atmosphere, the gage pressure is zero at both locations. The velocity at location 1 can be set equal to zero because of the large cross-sectional area in the container. By choice of datum,  $z_1 = H$  and  $z_3 = 0$ . Substituting all of this information into the Bernoulli Equation yields

$$0 + \frac{V_3^2}{2} + 0 = 0 + 0 + gH$$

or  $V_3 = \sqrt{2gH}$ , the same result that we obtained in Example 1 for the velocity out of the pipe at the bottom of a tank when the liquid level was located at a height  $H$  above the pipe. Of course, there would be some loss associated with the flow through the siphon pipe, which will lead to a smaller value of  $V_3$  than  $\sqrt{2gH}$ .

Now, let us find the gage pressure at location 2. We can write the Bernoulli Equation between locations 2 and 3.

$$\frac{p_3}{\rho} + \frac{V_3^2}{2} + gz_3 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Because the pipe has the same diameter throughout, the velocity  $V_2 = V_3$ . Substituting all the known information, we obtain

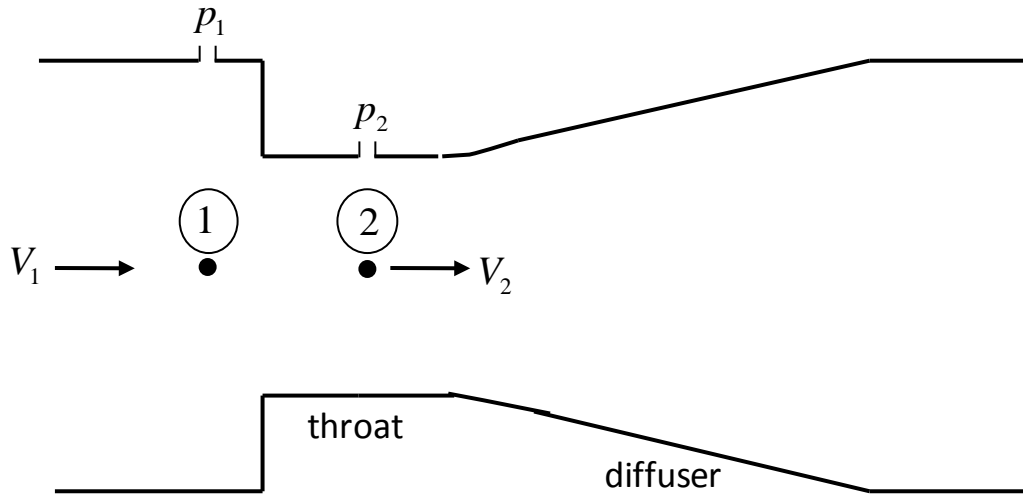
$$0 + \frac{V_3^2}{2} + 0 = \frac{p_2}{\rho} + \frac{V_3^2}{2} + g(H + h)$$

so that  $\boxed{p_2 = -\rho g(H + h)}$

This means that the pressure at location 2 is less than atmospheric pressure. Clearly there is a limit to how low we can go below atmospheric pressure. At first glance, you might think that we can increase the height difference between the lowest and highest points in the siphon pipe up to the value where the absolute pressure at location 2 will be zero. This would be an incorrect conclusion. Actually, when the pressure is gradually lowered in a liquid, it will first reach a value where it equals the vapor pressure at room temperature. When it goes slightly below this value, vapor bubbles will begin to form, typically at locations on the pipe wall that contain crevices with trapped air, a process known as heterogeneous nucleation. When vapor bubbles are nucleated, we say that “cavitation” is occurring in the liquid. The presence of such vapor bubbles will cause problems with operating the siphon, interfering with the flow.

Cavitation can be a serious problem in machinery in which the pressure drops below atmospheric pressure, perhaps because of a high velocity being reached at certain locations. The vapor bubbles that form will collapse when the liquid moves to a different location where the pressure is higher. Such collapsing of vapor bubbles can be violent and can cause pitting and erosion of material on the blades of centrifugal pumps, and on propeller blades used on ships.

### Example 3: Venturimeter



A venturimeter is a device that is placed in a pipe to permit the measurement of the flow rate of the fluid through the pipe. The measurement device is named in honor of Giovanni Venturi. The principle of the venturimeter is simple. The cross-sectional area of the flow is reduced suddenly to form a throat region. As the fluid accelerates to a higher velocity through the throat to accommodate the constancy of the volumetric flow rate, the kinetic energy must increase, and this must be accompanied by a drop in pressure. This pressure change is measured using pressure taps placed on the wall of the tube upstream of the constriction and in the throat. There are some losses associated with the contraction, but usually the largest losses accompany a sudden expansion in cross-sectional area that creates large recirculating eddies right after the expansion. These losses are avoided in the venturimeter by having the cross-section gradually expand through a “diffuser” section to the original pipe diameter. A well-designed venturimeter has minimal losses, and most of the pressure-drop is recovered downstream.

Given the fact that locations 1 and 2 in the schematic are at the same elevation, regardless of our choice of datum,  $z_1 = z_2$ . In the absence of any shaft work, and neglecting losses, the Bernoulli Equation reduces to

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} = \frac{p_1}{\rho} + \frac{V_1^2}{2}$$

Therefore, we can write  $V_2^2 - V_1^2 = 2 \frac{\Delta p}{\rho}$ , where  $\Delta p = p_1 - p_2$ . From conservation of mass,

$V_1 A_1 = V_2 A_2$ . We know that the areas are proportional to the square of the two diameters at locations 1 and 2. Therefore, we may write  $V_1 = \beta^2 V_2$ , where  $\beta = D_2 / D_1$ , the ratio of the diameters. Using this information, we can obtain

$$V_2 = \left[ \frac{2 \Delta p}{\rho (1 - \beta^4)} \right]^{1/2}$$



The volumetric flow rate through the pipe is then obtained from  $Q = \frac{\pi}{4} D_2^2 V_2$ , and the mass flow rate, if desired, as  $\rho Q$ . In spite of the careful design to minimize the losses, there still are some, and this fact is accommodated by multiplying the result for  $V_2$  by a discharge coefficient  $C_d$  which is a function of the diameter ratio  $\beta$ . The discharge coefficient is less than unity, but typically lies between 0.9 and 1.0.

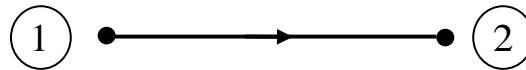
The venturimeter is relatively expensive, but minimizes operating costs by keeping the losses small. At the other end of the spectrum is the inexpensive orifice-meter, which involves placing an orifice-plate in the path of the flow. This plate contains a small orifice. The fluid accelerates through the orifice, and expands to fill the pipe again as it flows downstream. As noted earlier, in the sudden expansion of the cross-section past the orifice, recirculating eddies are formed, and the energy to maintain these eddies comes from the pressure-head. Therefore, while the principles of operation of both the venturimeter and the orifice meter are the same, the losses are much larger in the orifice meter. Modern orifice meters are relatively inexpensive and easy to install, but add to the pumping cost because of the losses.

The losses are usually reported as Non-recoverable head loss, measured in terms of throat velocity heads. The non-recoverable head loss for a venturimeter is typically of the order of a quarter throat velocity head, whereas that for orifice meters can range from 0.5 to 2.5 throat velocity heads, depending on the diameter ratio  $\beta$ .

### Example 4: Pitot-Static Tube

The Pitot-Static tube or simply Pitot tube, named after Henri Pitot, is a device that is employed to determine a local velocity within a flow field. It is based on the idea that when fluid moving at a certain velocity is brought to a stop at a stagnation point, its pressure rises. From the increase in pressure, we can infer the value of the velocity.

Consider points 1 and 2 in the sketch, and assume that point 2 is a stagnation point; that is, the velocity at point 2,  $V_2 = 0$ .



Because both points are at the same elevation, regardless of datum,  $z_1 = z_2$ . With no shaft work and negligible losses, the Bernoulli Equation reduces to

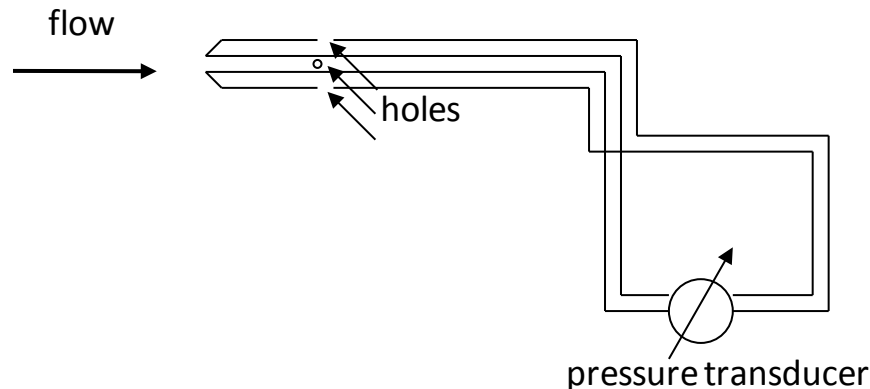
$$\frac{p_2}{\rho} + \frac{V_2^2}{2} = \frac{p_1}{\rho} + \frac{V_1^2}{2}$$

or 
$$p_2 = p_1 + \frac{\rho V_1^2}{2}$$

the pressure  $p_2$  is called the **stagnation pressure** for this streamline. We can solve the above equation for the velocity  $V_1$ .

$$V_1 = \sqrt{\frac{2(p_2 - p_1)}{\rho}}$$

This is the principle on which the Pitot tube is based. A typical Pitot tube consists of two hollow tubes of small diameter, one inside the other, as shown in the sketch adapted from White (1).



The annulus formed by the two tubes is closed off at the end facing the flow, and a few holes on the cylindrical surface of the outer tube permit the fluid to enter as the fluid flows past the Pitot tube in order to sample the pressure in the fluid flowing past the annulus. Therefore, the pressure of the fluid entering the annulus is the local pressure in the flow. The end of the inner hollow tube is connected to one side of a differential pressure transducer, so that the fluid has to come to a stop within that tube, raising the pressure to the stagnation pressure. The end of the annulus is connected to the other side of the same pressure transducer. Thus the pressure difference  $\Delta p = p_2 - p_1$  measured by the transducer can be used in the equation we obtained earlier to calculate the velocity. The instrument is quite small in diameter compared with the length scale on which the velocity varies in the flow, so that it can be assumed to measure the local velocity at the point where the sampling of the flow is performed.

The Pitot tube is employed on airplanes in flight to measure the speed of the plane, but a correction has to be applied for compressibility effects, because of the relatively large velocities involved. It also can be used to measure the local velocity of a liquid, such as that of blood in arteries and veins (1).

## References

1. F.M. White, Fluid Mechanics, Seventh Edition, McGraw-Hill, New York (2011).
2. M.M. Denn, Process Fluid Mechanics, Prentice-Hall, Englewood Cliffs (1980).