LARGE AMPLITUDE PERIODIC WAVES BENEATH AN ICE SHEET

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ABSTRACT
In this paper we look at two-dimensional, periodic waves of large amplitude propagating beneath an elastic ice sheet floating on an infinitely deep, inviscid fluid. Approximate solutions are found using the method of Forbes (1988) as a Fourier series whose coefficients are computed numerically. Several different large amplitude (nonlinear) models for the sheet are considered and each is compared to the standard (linear) Euler-Bernoulli model.

INTRODUCTION
The vast sheets of sea ice that cover the polar oceans are continuously subjected to cyclic stresses associated with wave action that may have originated in the open sea abutting the ice or may have been generated in the ice cover itself by the wind. Observations suggest that the so-called ice-coupled wave spectrum is an important and omnipresent feature of the Arctic ice cover, for example, of frozen bays and harbours, and of the pack ice mosaic in the marginal ice zone (MIZ). Its presence alters the morphology of the ice in various ways, notably, it causes cracks—thereby enhancing break-up, it may fatigue the ice (Langhorne et al., 2001), and it redistributes the floes that make up the ice veneer.

Considerable work has been done since the first theoretical model of wave propagation in sea ice was proposed by Greenhill (1897). A number of field programmes have taken place to collect data in situ, and theoretical work continues to this day. However, a common thread in nearly all of this work is the assumption that the ice sheet can be modelled as a thin, Euler-Bernoulli elastic plate, resting on an inviscid fluid foundation and that the deformation gradients experienced by the ice due to the waves are of sufficiently small amplitude that a linearized analysis may be employed. Few papers allow the ice to be thicker or, what is the subject of this paper, large deformation gradients to be considered. Yet, particularly near the ice margin, waves can be of high amplitude and the deflections seen in the sea ice can be substantial. Moreover, large deformation events have occasionally been experienced hundreds of kilometres from the ice edge (Marko, 2002).

The goal of the current paper is to examine how the properties of ice-coupled waves are affected by the choice of elastic plate model used to represent the behaviour of sea ice.
While it is recognized that sea ice is inelastic, viscosity is not included in the treatment (this enhancement would present little difficulty) so that each model considered is perfectly elastic. The investigation focuses instead on nonlinear effects brought about by large deformation gradients. Four separate plate models are considered and are dealt with in turn: (i) the Euler-Bernoulli thin plate, which is used as the control; (ii) The Forbes model (Forbes, 1986, 1988), which uses the correct expression for the Gaussian curvature instead of the usual linearized version; (iii) the classical von Karman plate (Fung, 1965), which introduces an additional axial strain that arises because of the incorporation of a nonlinear term in the strain tensor; and (iv) the Drozdov plate (Drozdov, 1996), which is similar to the von Karman plate but uses a revised expression for the axial strain. No spatial variation in the material properties is assumed to occur through the ice thickness, which is equivalent to assuming that integrated moduli are used in the manner of Kerr and Palmer (1972). In each case the same method of solution is used, which is due to Forbes (1986, 1988). The phase speeds of the waves are compared at physically plausible wave amplitudes, and the shape of the wave profiles is plotted in each case. The amplitudes considered by Forbes (1986, 1988) are much larger than those examined here; his work focuses more on the interesting mathematical structure that can occur, e.g. bifurcation, rather than on geophysical aspects. For this reason we find that the nonlinearity introduced when Gaussian curvature is used is negligible compared to the nonlinearity that arises due to axial strain.

FORMULATION

We consider steady progressive surface waves of (peak to trough) height $2A$, speed $c$, and wavelength $\lambda$ in a fluid of infinite depth and density $\rho$ beneath an elastic sheet of thickness $T$ and density $\rho_M$, as in Figure 1.

![Figure 1: The fluid-plate.](image)

The problem is nondimensionalized by scaling all lengths by $2\pi/\lambda$ and velocities by $(2\pi/g\lambda)^{1/2}$. This makes the wavelength $2\pi$, the sheet thickness $H = 2\pi T/\lambda$ and the half wave height $\alpha = 2\pi A/\lambda$. The bending moment of the beam is scaled by $(2\pi)^3/(\rho_M g \lambda^3)$ and the pressure by $2\pi/(\rho g \lambda)$. We superimpose on the wave profile a (non-dimensional) coordinate frame moving at velocity $\mu^{1/2}$ with the wave, where the (non-dimensional) $x$-axis is aligned along the equilibrium surface position and the $y$-axis is perpendicular to this, through the wave peak.

In terms of the reference frame moving with the wave, the horizontal and vertical components of velocity may be expressed in terms of the velocity potential $\phi(x, y)$ and stream function $\psi(x, y)$, which (since $\phi$ is harmonic) satisfy the Cauchy-Riemann equations, as...
follows:

\[ \phi_x = \psi_y, \quad \phi_y = -\psi_x \]  

and also the asymptotic conditions

\[ \phi_x \to -\mu^{1/2}, \quad \phi_y \to 0 \text{ as } y \to -\infty. \]  

The surface of the fluid satisfies Bernoulli’s equation

\[ \frac{1}{2}(\phi_x^2 + \phi_y^2) + y + P = \frac{\mu}{2} + d_M H. \]  

where \( \mu = 2\pi c^2/(g\lambda) \) is the non-dimensional wave speed and \( d_M = \rho_M / \rho \). The pressure \( P \) is the pressure exerted on the surface by the plate. This may be given by any of the plate models described below.

**Euler-Bernoulli model**

Assuming the displacement, slope and curvature are infinitesimally small, we can model the pressure using the standard (linear) Euler-Bernoulli equation

\[ P = d_M (H - M_{xx}), \quad -\pi < x < \pi, \]  

where the bending moment \( M \) is given by \( M = -Kv_{xx}, \quad K = 16\pi^4 ET^3/(12\rho_M g \lambda^4(1 - \nu^2)) \). Under these assumptions, we can also eliminate the nonlinear terms in the Bernoulli equation for the surface, leaving

\[ \nu + d_M K v_{xxxx} = \mu \nu. \]

This, coupled with the condition that the wave height is \( 2\alpha \), can be solved to give the linear solution

\[ \nu(x) = \alpha \cos x, \quad \mu = 1 + d_M K. \]  

Note that, in contrast to the nonlinear models that follow, the wave speed is independent of the wave height.

**The Forbes model**

To account for large (finite) curvatures Forbes (1986) proposed using the same formula for pressure as in (4) but with the bending moment given in terms of the Gaussian curvature, adjusted because of the position of the origin, as follows:

\[ M = \frac{-Kv_{xx}}{(1 + v_x^2)^{3/2} - Hv_{xx}/2} \]

instead of \( M = -Kv_{xx} \), as in the linear model.

**The von Karman model**

The Euler-Bernoulli plate model is formulated under the assumption that the displacements are infinitesimal and it is inaccurate for displacements of the order of the thickness of the plate. The formulation of the von Karman plate model starts out in much the same way as the Euler-Bernoulli model, defining the displacements as

\[ u_1 = u - yv_x, \quad u_2 = v, \]
where $u = u(x)$ and $v = v(x)$ are the horizontal and vertical displacements of the centre of the plate respectively. In this case, however, the strains are calculated using the Green strain tensor

$$E_{ij} = \frac{1}{2} \left( \frac{\partial u}{\partial a_j} + \frac{\partial u}{\partial a_i} + \frac{\partial u}{\partial a_i} \frac{\partial u}{\partial a_j} \right), \quad a_1 = x, a_2 = y,$$

the only significant term in which is the axial strain

$$E_{11} = u_x - yv_{xx} + \frac{1}{2}v_{x}^{2}.$$

The axial stress can be derived from this using Hooke’s law, from which we can in turn derive the resultant stress $N$ and bending moment $M$. The pressure equation in this case may be written

$$P = d_{M}(H - Nv_{xx} + Kv_{xxxx}), \quad (8)$$

where

$$N = \frac{L}{4\pi} \int_{-\pi}^{\pi} v_{x}^{2} dx, \quad L = \frac{4\pi^{2}ET}{\rho_{M}g\lambda^{2}(1-\nu^{2})}. \quad (9)$$

**The Drozdov model**

This model for the plate is similar to a nonlinear rod model found in Drozdov (1996), which is itself a special planar case of the general nonlinear rod models described by Antman (1973). The model incorporates the nonlinear effects in both the Forbes and von Karman models while also accounting for the normal action of the fluid pressure upon the plate. The axial strain in this case is given by

$$\epsilon_{11} = u_x - y\kappa, \quad u = s - x,$$

where $\kappa$ is the Gaussian curvature and $s$ is the arc-length along the plate. Taking the derivative of $u$ with respect to $x$, we get

$$u_x = \frac{du}{dx} = \frac{ds}{dx} - 1 = \sqrt{1 + v_{x}^{2}} - 1.$$

We can calculate axial stress from this using Hooke’s law $\sigma_{11} = E\epsilon_{11}$ and from this the stress resultant and bending moment

$$N = \int_{-h/2}^{h/2} \sigma_{11} dy = Lu_x,$$

$$M = \int_{-h/2}^{h/2} y\sigma_{11} dy = -K\kappa.$$  

This gives us the formula for the pressure

$$P \cos \beta = d_{M} \left( H - \frac{\partial}{\partial x} \left( \frac{\partial M}{\partial x} \cos^{2} \beta - N \sin \beta \right) \right), \quad (10)$$

where $\beta$ is the angle the tangent to the plate makes with the vertical.
METHOD OF SOLUTION

The periodic solutions sought may be expressed as Fourier series in the form

\[ z(r, \theta) = x(r, \theta) + i y(r, \theta) = i \left[ \log \zeta + \sum_{j=1}^{\infty} A_j \zeta^j \right], \quad \zeta = re^{i\theta}. \quad (11) \]

where \( r = 1 \) corresponds to the fluid surface and \( r = 0 \) to \( y = -\infty \). These solutions are required to satisfy Bernoulli’s equation

\[ \frac{\mu}{2} \left( \frac{1}{x_\theta^2 + y_\theta^2} - 1 \right) + y + P = d_M H, \quad (12) \]

where \( x = x(1, \theta), y = y(1, \theta) \), and the constraint

\[ y(1, 0) - y(1, \pi) = 2\alpha. \quad (13) \]

To form an approximate solution truncate the Fourier series for \( x(\theta) = x(1, \theta) \) and \( y(\theta) = y(1, \theta) \) after order \( N \), i.e.

\[ x(\theta) = -\theta - \sum_{j=1}^{N} A_j \sin(j\theta), \quad (14) \]

\[ y(\theta) = A_0 + \sum_{j=1}^{N} A_j \cos(j\theta). \quad (15) \]

We get similar series for the derivatives \( x_\theta, y_\theta, x_{\theta\theta}, y_{\theta\theta} \) by differentiating these formulae with respect to \( \theta \). We also propose similar truncated Fourier series for the pressure at the surface

\[ P(\theta) = \sum_{j=0}^{N} P_j \cos(j\theta), \quad P_j = \frac{1}{\gamma_j \pi} \int_{-\pi}^{\pi} P(\theta) \cos(j\theta) d\theta, \quad (16) \]

where the \( P(\theta) \) in the integral comes from the respective plate model used, i.e. (8) for the von Karman plate etc., and also for the other nonlinear term \( R(\theta) = 1/(x_\theta^2 + y_\theta^2) \) in (12)

\[ R(\theta) = \sum_{j=0}^{N} R_j \cos(j\theta), \quad R_j = \frac{1}{\gamma_j \pi} \int_{-\pi}^{\pi} \cos(j\theta) x_\theta^2 + y_\theta^2 d\theta, \quad (17) \]

where \( \gamma_0 = 2 \) and \( \gamma_j = 1 \), for \( j \geq 1 \). All the integrals in the coefficients \( P_j, R_j \) may be easily estimated using the trapezoidal rule and hence written in terms of the unknowns \( \mu, A_0, ..., A_N \). Substituting these series into the equations (12) and (13), we get a system of \( N + 2 \) equations for the \( N + 2 \) unknowns \( \mu, A_0, ..., A_N \)

\[
\begin{align*}
\frac{\mu}{2}(R_0 - 1) + A_0 + P_0 &= d_M H, \\
\frac{\mu}{2}R_j + A_j + P_j &= 0, \quad j = 1, ..., N \\
\sum_{k=0}^{[(N-1)/2]} A_{2k+1} &= \alpha,
\end{align*}
\]

(18)

where \( [(N - 1)/2] \) is the integer part of \( (N - 1)/2 \). This system may then be solved numerically for the \( N + 2 \) unknowns \( \mu, A_0, ..., A_N \) using a method such as a damped Newton-Raphson, as described in Forbes (1988).
RESULTS

In this section the following values for the required physical parameters are used: $E = 5 \text{ GPa}$, $\nu = 0.3$, $\rho = 1025 \text{ kg m}^{-3}$, $\rho_M = 922.5 \text{ kg m}^{-3}$ and $g = 9.81 \text{ m s}^{-2}$, $T = 1 \text{ m}$, and $\lambda = 100 \text{ m}$.

In Figures 2 and 3 a comparison is made of the difference of the phase velocity for each nonlinear model $\mu$ and that for the linear, Euler-Bernoulli plate $\mu_0$. Because the change in $\mu - \mu_0$ due to the adoption of the Gaussian curvature—as opposed to its linearized form—is much smaller than the effect due to the incorporation of axial strain, the two figures cannot easily be combined. For the physically-realistic non-dimensional amplitudes shown, the use of a nonlinear curvature causes a gradual deviation in $\mu - \mu_0$ as $\alpha$ increases, as seen in Figure 2. By the time $\alpha$ has reached a value of 0.1, corresponding to a dimensional wave amplitude of 1 m, $\mu$ is only $4 \times 10^{-3}$ larger than the equivalent linear phase velocity. In Figures 3, however, very significant changes in the phase velocity are predicted for both the von Karman and Drozdov models, with the effect being similar for both. (This is not unexpected as it arises because of the inclusion of axial strain in the plate equation in both cases.) For waves of physical amplitude $A = 1.6 \text{ m}$ the phase velocity is about $33 \text{ m s}^{-1}$ (approximately twice the phase velocity for the linear model). One can easily justify the phase velocity for the von Karman model by assuming (as appears in Figure 4) that the displacement is approximately
\[ v(x) \approx \alpha \cos x, \text{ and consequently } N \approx \alpha^2 L/4. \]

From this we get

\[ \mu - \mu_0 \approx \frac{\alpha^2 dM L}{4}, \]

which is about 5.5 at \( \alpha = 0.1 \), approximately agreeing with the von Karman curve in Figure 3.

The wave profiles shown in Figure 4 illustrate that the nonlinearities considered have negligible effect on the shape of the ice-coupled wave of physical amplitude \( A = 1.6 \) m. The profiles do, however, differ significantly from the linear wave profile at larger amplitudes, although such larger amplitudes are physically unrealistic because they cause the ice plate to experience larger strains than it can sustain.

**DISCUSSION AND CONCLUSIONS**

The modelling done in this paper suggests two important results: (i) the use of Gaussian curvature, as opposed to its linearized form, has negligible effect on the ice-coupled wave profile or its phase velocity; (ii) the inclusion of axial strain in the plate equation, either by using a von Karman or a Drozdov formulation, significantly alters the character of ice-coupled waves when their amplitude is large but their slopes are such that they are still physically possible.

The outcome that axial strain should be included in the plate equation when larger wave amplitudes are modelled is not entirely surprising. It suggests that simple, linear (Euler-Bernoulli) models of ice deformation will not be accurate when, for example, intense wave-in-ice events are described theoretically, such as that experienced by Marko (2002) in the Sea of Okhotsk or by one of the current authors (VAS) in the Weddell Sea. No in situ field data exist to validate our conclusions, as such events invariably lead to destruction of the ice cover and any equipment that is in place at the time, but laboratory experiments could be done in an appropriately equipped wave flume.
ACKNOWLEDGEMENT
This work described was supported by a Marsden grant from the Royal Society of New Zealand and by the Foundation for Research, Science and Technology.

REFERENCES