EE/ME/AE324: Dynamical Systems

Chapter 8: Transfer Function Analysis

The System Transfer Function

- Consider the system described by the *n*th-order I/O eqn.:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_m u^{(m)} + \cdots + b_0 u$$

- Taking the Laplace transform of the system eqn. with $\text{ICs} = 0$:

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_0\right)Y(s) = \left(b_m s^m + \cdots + b_0\right)U(s)$$

- The Transfer Function (TF) is defined as:

$$H(s) \triangleq \left.\frac{Y(s)}{U(s)}\right|_{ICs=0} = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

- Factoring the TF yields:

$$H(s) = K\left[\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}\right]$$

where p_i and z_i are the system poles and zeros, respectively

The System Transfer Function

- If the PFE of the TF has the form:

$$H(s)=\frac{A_1}{(s-p_1)}+\frac{A_2}{(s-p_2)}+\cdots+\frac{A_n}{(s-p_n)}$$

where A_i are the residues associated with the system poles, the zero-input system response will have the form:

$$y_{zi}(\mathrm{t})=K_1e^{p_1t}+K_2e^{p_2t}+\cdots+K_ne^{p_nt}$$

where the e^{p_it} terms are called the system modes

- The stability of the system response is based on the p_i:

$\Rightarrow$ Stable if $\Re\{p_i\}<0$ for all p_i

$\Rightarrow$ Unstable if $\Re\{p_i\}>0$ for any p_i

$\Rightarrow$ Marginally stable (oscillatory) if $\Re\{p_i\}=0$ for distinct p_i

Second Order Responses

- Assume a 2nd order TF of the form

$$H(s) \triangleq \frac{Y(s)}{U(s)} = \frac{G_{DC}\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}} = \frac{G_{DC}\omega_n^{\ 2}}{\Delta(s)}$$

where $0 < \zeta < 1$ is the damping ratio (unitless), G_{DC} is the DC gain and ω_n is the natural frequency [rad/sec]

- The characteristic poly. $\Delta(s)$ can be factored as

$$\Delta(s) = \left(s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)\left(s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)$$

$$= \left(s + \frac{1}{\tau} + j\omega_d\right)\left(s + \frac{1}{\tau} - j\omega_d\right)$$

Second Order Responses

- This implies that the roots (poles) of $\Delta(s)$ are:

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\frac{1}{\tau} \pm j\omega_d$$

where $\tau \triangleq \dfrac{1}{\zeta\omega_n}$ is the time constant [sec] and

$\omega_d \triangleq \omega_n\sqrt{1-\zeta^2}$ is the damped frequency [rad/sec]

- It also implies $\left|s_{1,2}\right| = \sqrt{\zeta^2\omega_n^{\,2} + \omega_n^{\,2}\left(1-\zeta^2\right)} = \omega_n$ is the distance from the complex poles to the origin of the s-plane, assuming $|\zeta| < 1$

Second Order Step Responses

- We now visualize second order system responses to unit step inputs for $G_{DC} = \omega_n = 1$ as ζ varies

- Note, $Y(s) = H(s) \cdot U(s) = \dfrac{G_{DC}\omega_n{}^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n{}^2\right)}$

$$\rightleftharpoons y(t) = G_{DC} \cdot \left[1 - \left(\frac{1}{\sqrt{1-\zeta^2}}\right) e^{-\left(t/\tau\right)} \sin\left(\omega_d t - \phi\right)\right],$$

$$\text{where } \phi = \tan^{-1}\left(\frac{-\sqrt{1-\zeta^2}}{\zeta}\right) \text{ for } 0 < \zeta < 1$$

Second Order Step Responses

- In the plots that follow for $\left|\zeta\right| \leq 1$:

$$\%\text{Overshoot}\left(\zeta\right) = 100 \cdot \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

$$\Rightarrow \%\text{OS}\left(0\right) = 100\%,\ \ \%\text{OS}\left(.5\right) = 16.3\%,$$

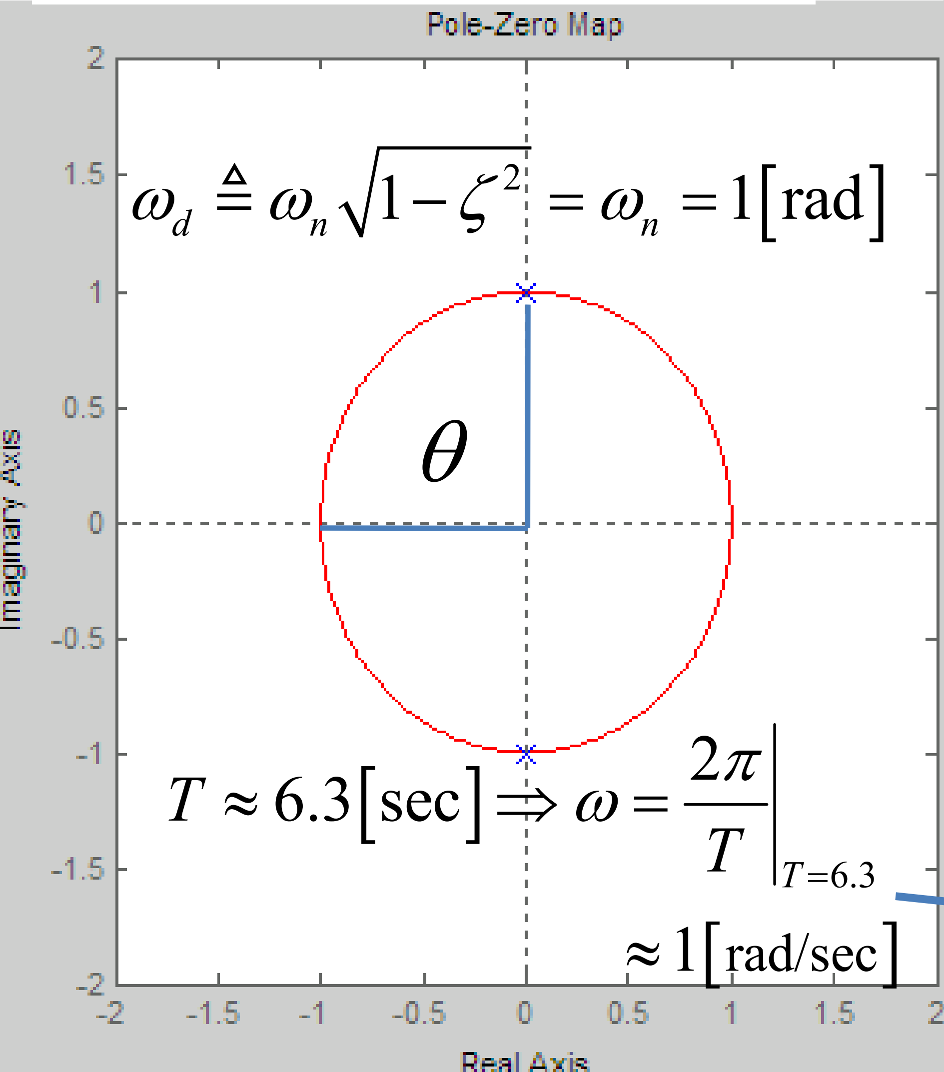
$$\%\text{OS}\left(.707\right) = 4.3\%,\ \ \%\text{OS}\left(1\right) = 0\%$$

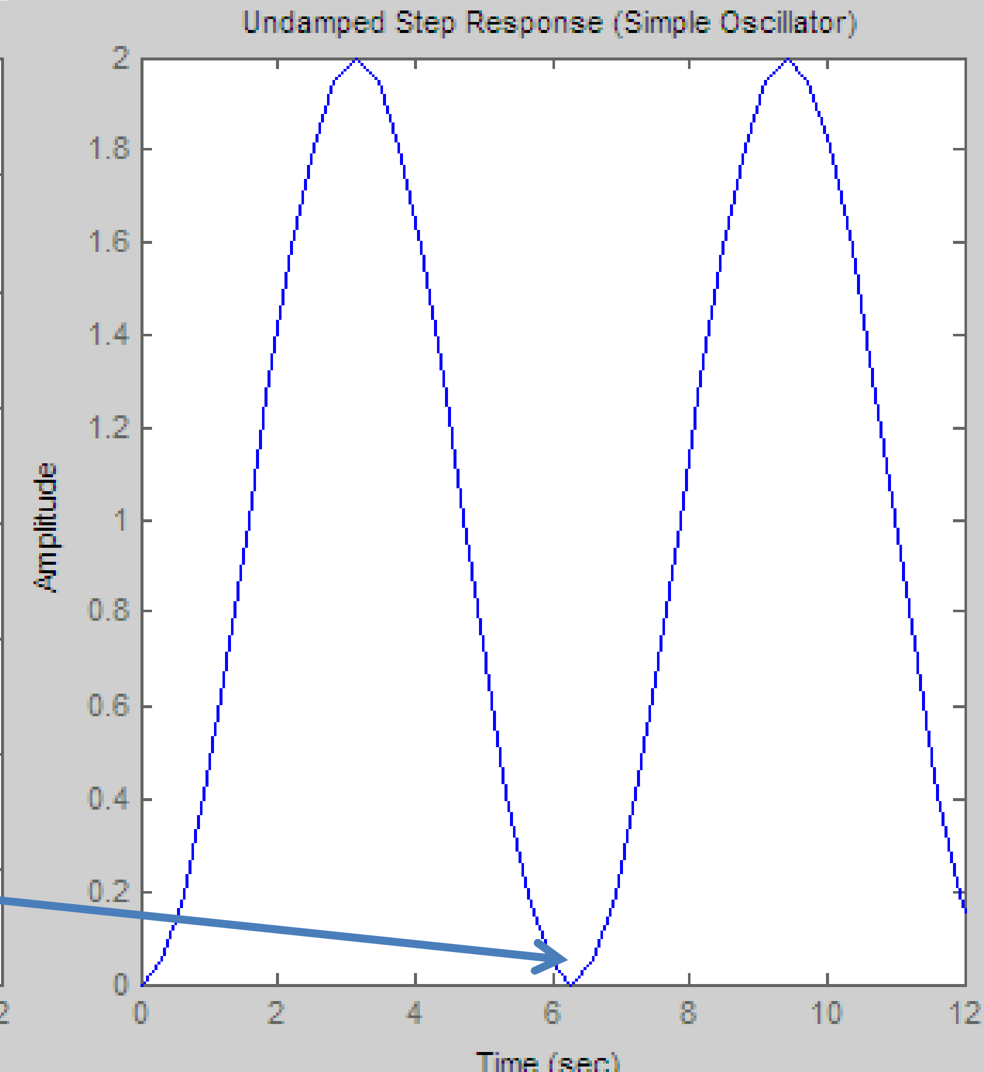
- Second_Order_Response.m on the class web site will be used to generate pole-zero plots and step responses

```
Transfer function:
   1
-------
s^2 + 1
```

$\Rightarrow \zeta = 0$, marginally stable (undamped) sys.

$$\theta = \cos^{-1}(\zeta) = 90^\circ, \text{ valid only for } |\zeta| \le 1$$

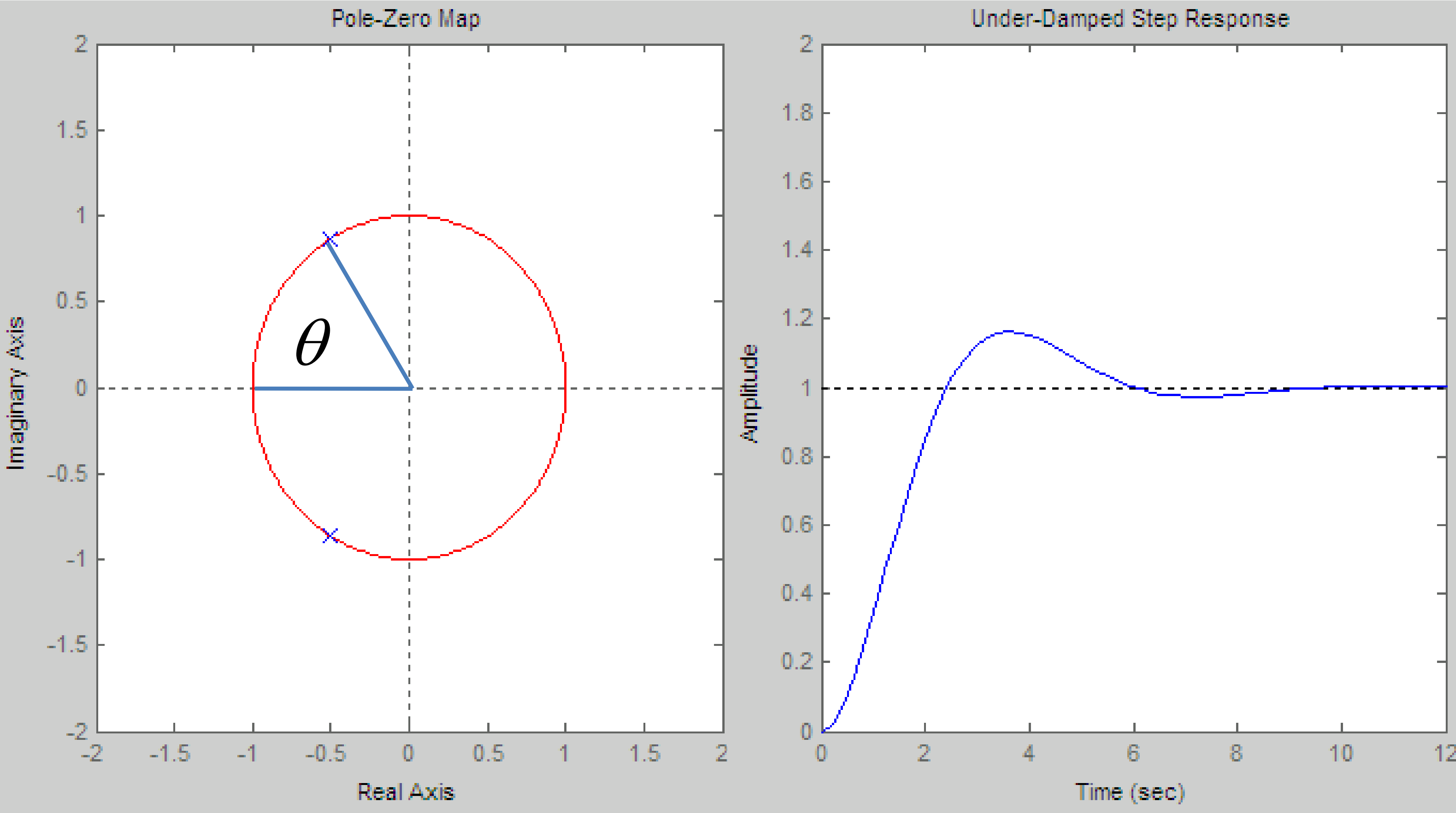




```
Transfer function:
      1
-----------
s^2 + s + 1
```

$$\Rightarrow \zeta = 0.5\text{, under-damped (stable) sys.}$$

$$\theta = \cos^{-1}(\zeta) = 60^\circ$$

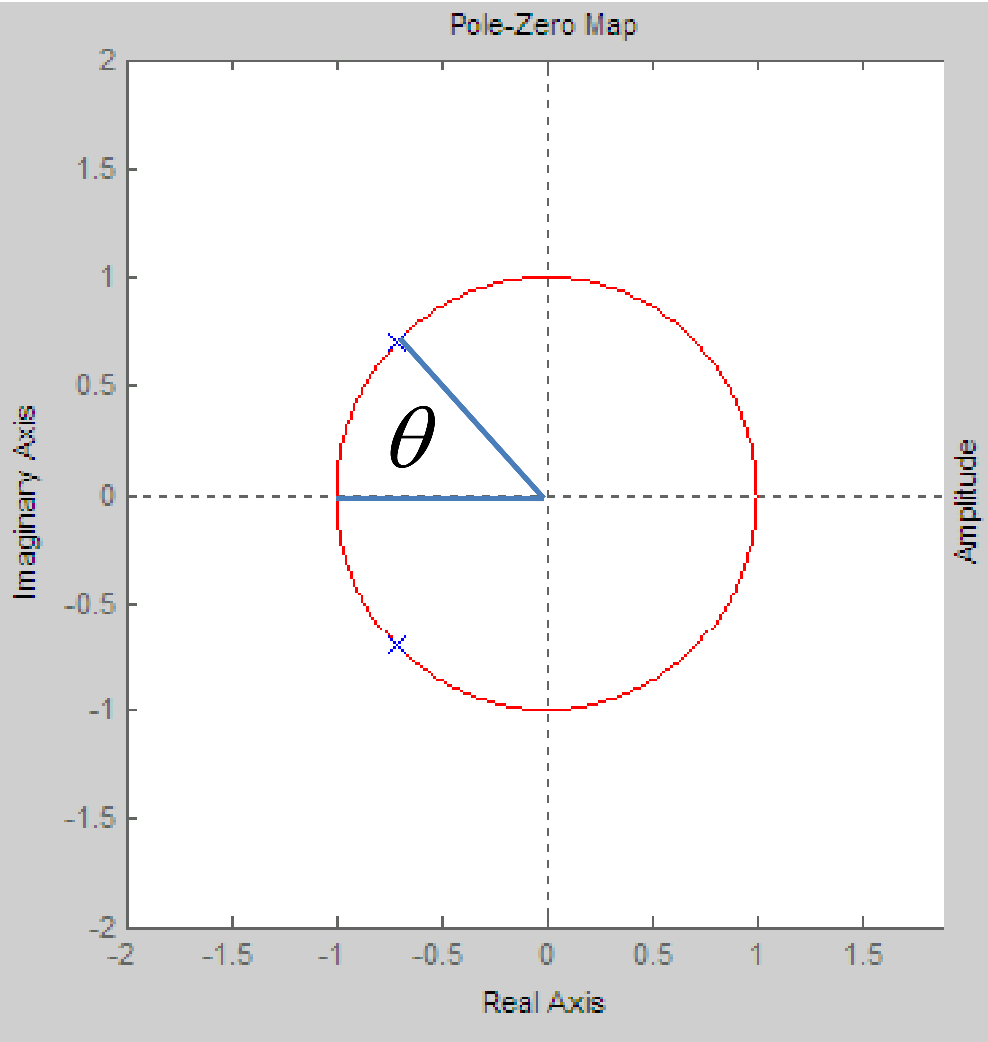


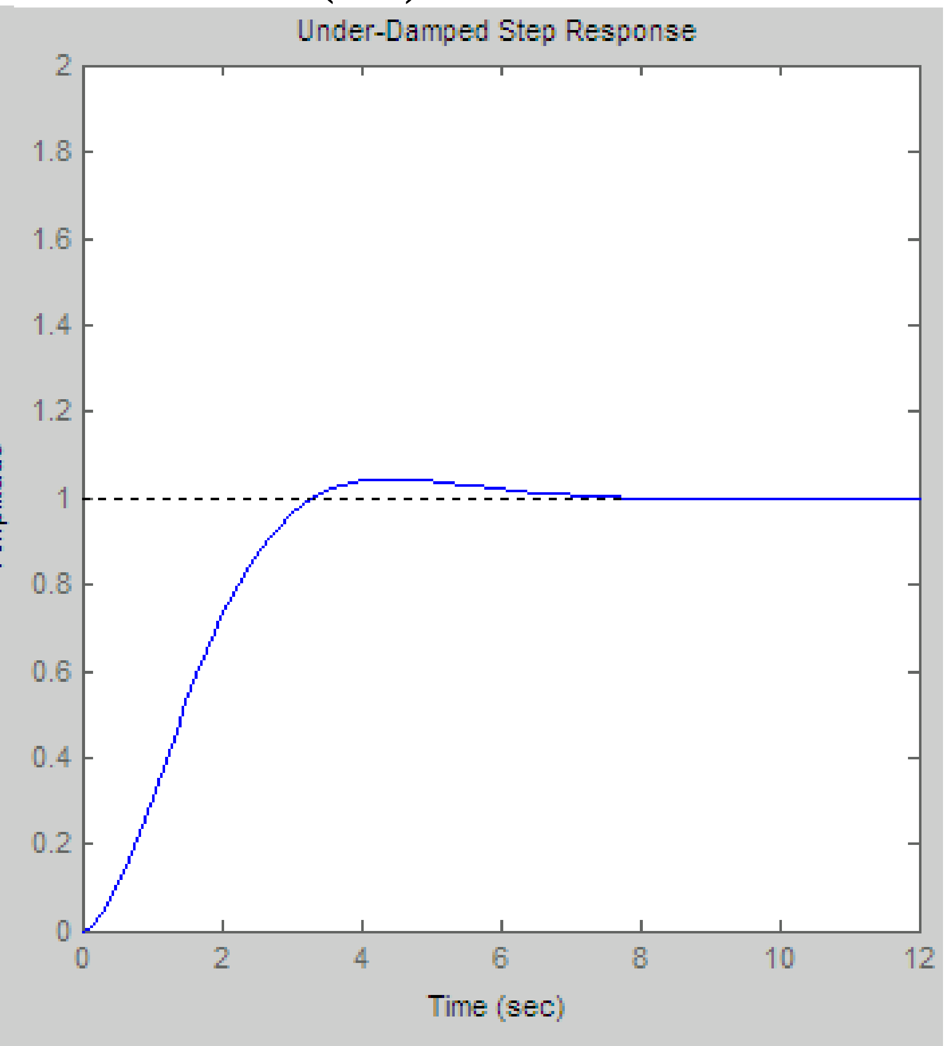
```
Transfer function:
        1
-----------------
s^2 + 1.414 s + 1
```

$$\Rightarrow \zeta = 0.707,$$

under-damped (stable) sys.

$$\theta = \cos^{-1}(\zeta) = 45^\circ$$

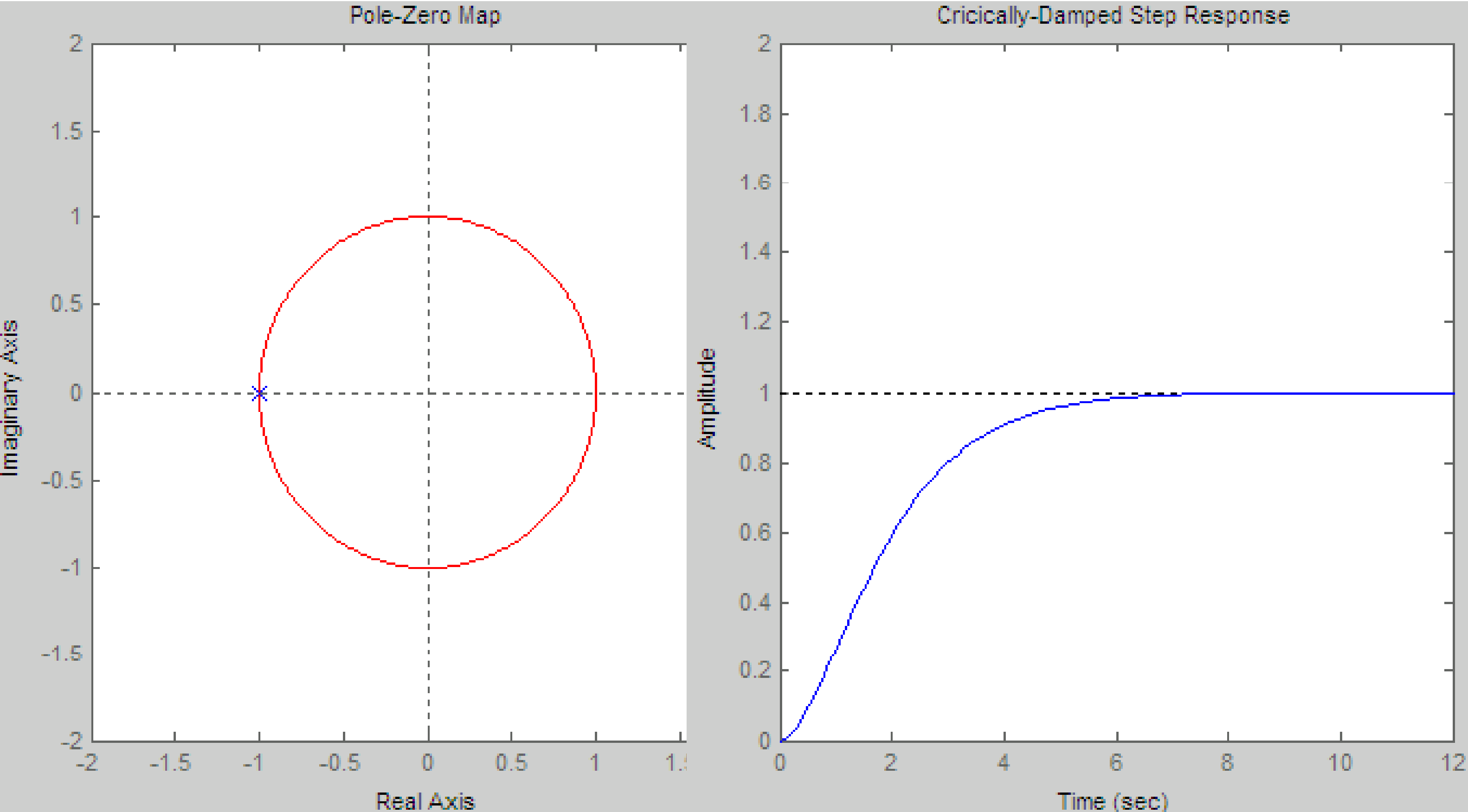




```
Transfer function:
       1
-------------
s^2 + 2 s + 1
```

$$\Rightarrow \zeta = 1\text{, critically damped (stable)}$$

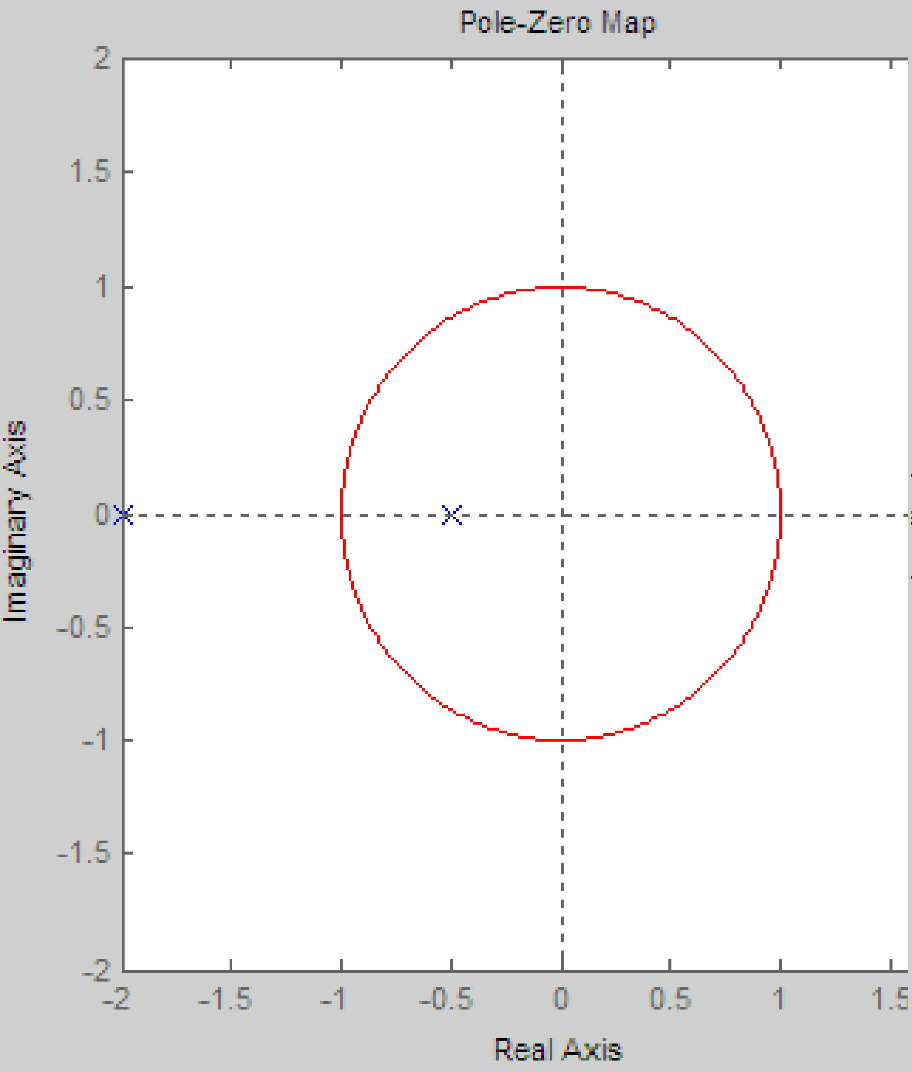
$$\theta = \cos^{-1}(\zeta) = 0^\circ$$

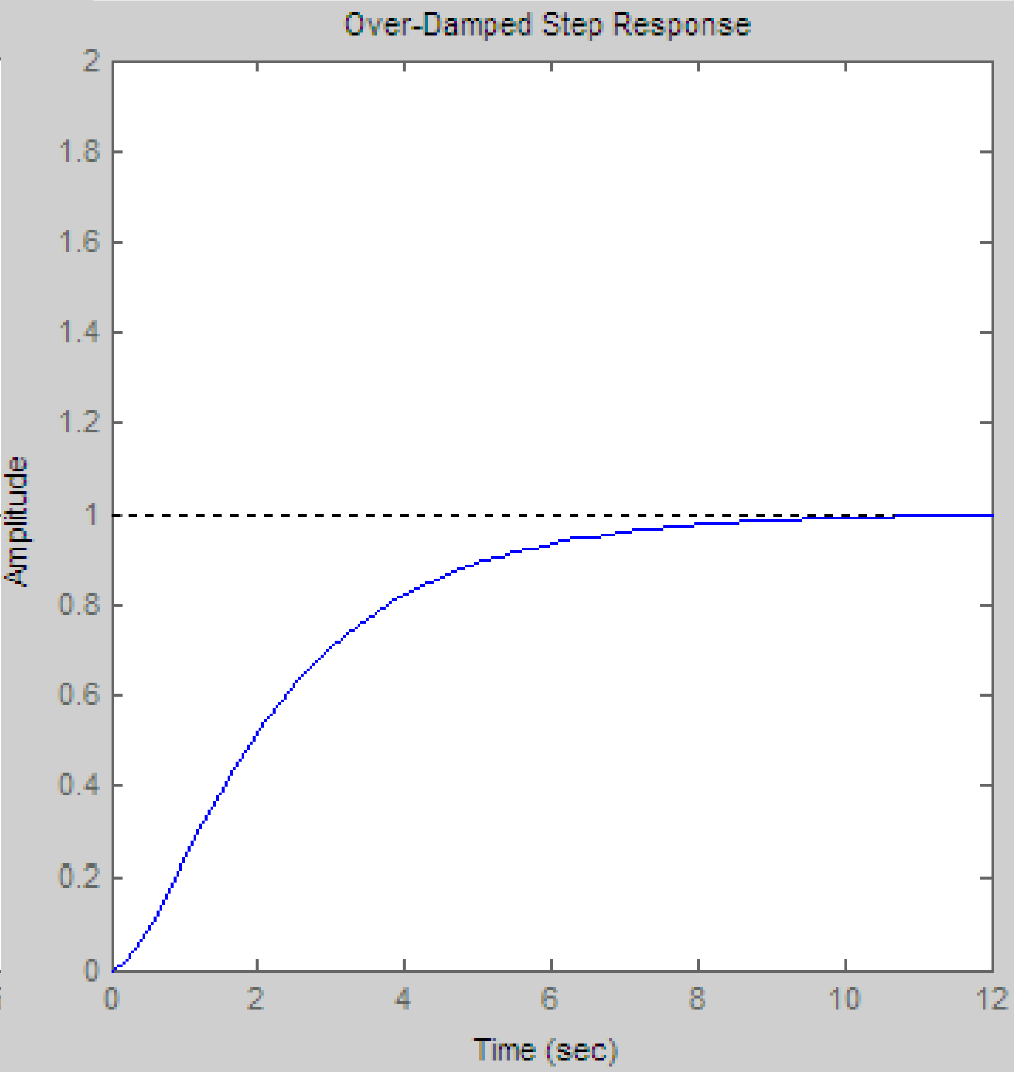


```
Transfer function:
        1
---------------
s^2 + 2.5 s + 1
```

$\Rightarrow \zeta = 1.25,$

over-damped (stable) sys.





The Frequency Response

- We now consider the steady-state response of a general system to a sinusoidal input of the form:

$$u(t) = \sin(\omega t) \rightleftharpoons U(s) = \left[\frac{\omega}{s^2 + \omega^2}\right]$$

- The system output can be expressed using its TF as:

$$Y(s) = H(s) \cdot U(s) = H(s)\left[\frac{\omega}{s^2 + \omega^2}\right]$$

$$= H(s)\left[\frac{\omega}{(s + j\omega)(s - j\omega)}\right]$$

The Frequency Response

- Performing PFE, we have: $Y(s) = \dfrac{C}{s - j\omega} + \dfrac{C^*}{s + j\omega}$

  $+\left[\text{terms associated with the stable poles of } H(s) \right]$

- The residue C can be evaluated as shown in Chapter 7, e.g.: $C = \left(s - j\omega\right) Y(s)\Big|_{s=j\omega}$

$$= H(s)\left[\frac{\omega}{s + j\omega}\right]\Bigg|_{s=j\omega} = \frac{H\left(j\omega\right)}{j2}$$

The Frequency Response

- Define $H(j\omega) \triangleq |H(\omega)| \measuredangle H(\omega) = M(\omega)\mathrm{e}^{j\theta(\omega)}$, where $M(\omega)$ and $\theta(\omega)$ are real functions representing the magnitude and phase angle of $H(j\omega)$ respectively; then $C^* = -\frac{H(-j\omega)}{j2} = -\frac{M(\omega)\mathrm{e}^{-j\theta(\omega)}}{j2}$
- Since the poles of $H(s)$ are assumed stable, the steady-state response of these poles is zero implying

$$Y_{SS}(s) = \frac{M(\omega)\mathrm{e}^{j\theta(\omega)}}{j2(s-j\omega)} - \frac{M(\omega)\mathrm{e}^{-j\theta(\omega)}}{j2(s+j\omega)}$$

The Frequency Response

$$\Rightarrow Y_{SS}(s) = \frac{M(\omega)}{j2}\left[\frac{\mathrm{e}^{j\theta(\omega)}}{(s - j\omega)} - \frac{\mathrm{e}^{-j\theta(\omega)}}{(s + j\omega)}\right] \rightleftharpoons$$

$$y_{SS}(t) = M(\omega)\left[\frac{e^{j\left(\omega t+\theta(\omega)\right)} - e^{-j\left(\omega t+\theta(\omega)\right)}}{j2}\right]$$

$$= M(\omega)\sin\left(\omega t + \theta(\omega)\right)$$

- $H(j\omega) = M(\omega)\measuredangle\theta(\omega)$ is known as the system's frequency response function (or frequency response)

The Frequency Response

- Using a similar analysis, it can be shown that inputs of the form: $u(t) = A\sin(\omega t + \phi)$ and

$$A\cos(\omega t + \phi)$$

produce steady-state outputs of the form:

$$y_{SS}(t) = AM(\omega)\sin\left(\omega t + \phi + \theta(\omega)\right) \text{ and}$$

$$AM(\omega)\cos\left(\omega t + \phi + \theta(\omega)\right), \text{ respectively}$$

The Frequency Response

- This implies that what comes out of a linear system is simply a scaled and shifted verion of what is input;

$$\Rightarrow \text{If } u(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\omega_0 nt + \phi_n),$$

$$= \text{DC Component} + \text{Fundamental frequency } \omega_0 \text{ and Higher order harmonics}$$

then

$$y_{SS}(t) = A_0 M(0) + \sum_{n=1}^{\infty} A_n M(\omega_0 n) \cos\left(\omega_0 nt + \phi_n + \theta(\omega_0 n)\right)$$

- This is reason why Fourer Series are commonly used to analyze systems and signals (not cover in this class)!

A Frequency Response Example

- Find $y_{SS}(t)$ when the stable system defined by

$$H(s) = \frac{4s+8}{s^2+2s+5}$$ is driven by the input

$$u(t) = 5\sin(4t + \frac{\pi}{3})$$

- Evaluating the TF at the input frequency yields

$$M(\omega) = \left|H(s)\right|\Big|_{s=j\omega=j4} = \left|\frac{4s+8}{s^2+2s+5}\right|\Bigg|_{s=j\omega=j4}$$

$$= 1.3152$$ $>> abs((4*j*4+8)/((j*4)^2+2*j*4+5))$

A Frequency Response Example

and

$$\theta(\omega) = \measuredangle H(s)\Big|_{s=j\omega=j4} = \measuredangle\left[\frac{4s+8}{s^2+2s+5}\right]\Bigg|_{s=j\omega=j4}$$

$$= -1.4056 \text{ [rad]}$$

$$>> angle((4*j*4+8)/((j*4)^2+2*j*4+5))$$

$$\Rightarrow y_{SS}(t) = 5M(\omega)\sin\left(4t+\frac{\pi}{3}+\theta(\omega)\right)$$

$$= 6.576\sin(4t-0.3585)$$

A Frequency Response Example

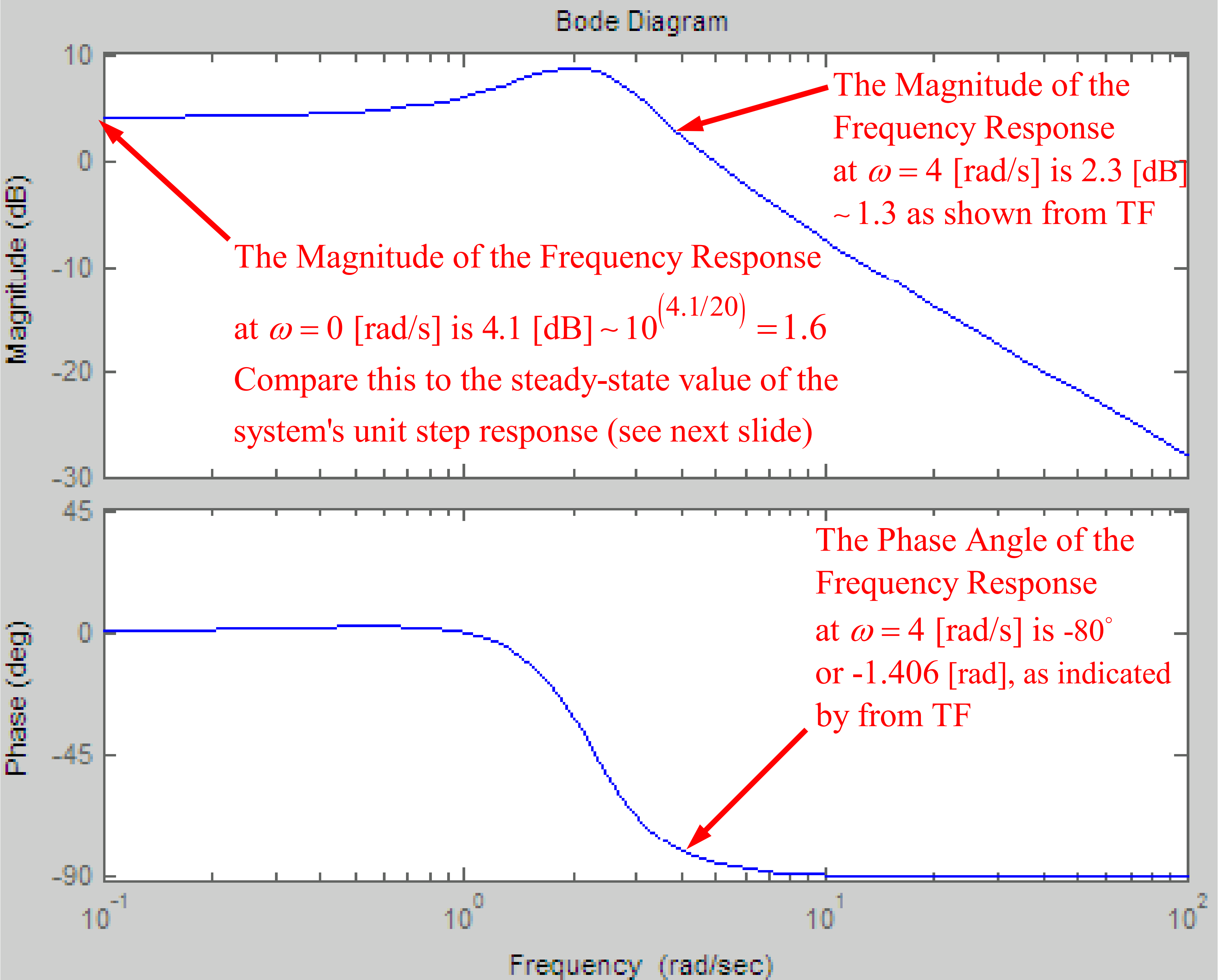
- We can predict how this system will pass any input frequency by plotting $M(\omega)$ and $\theta(\omega)$ using the Matlab bode command as shown:

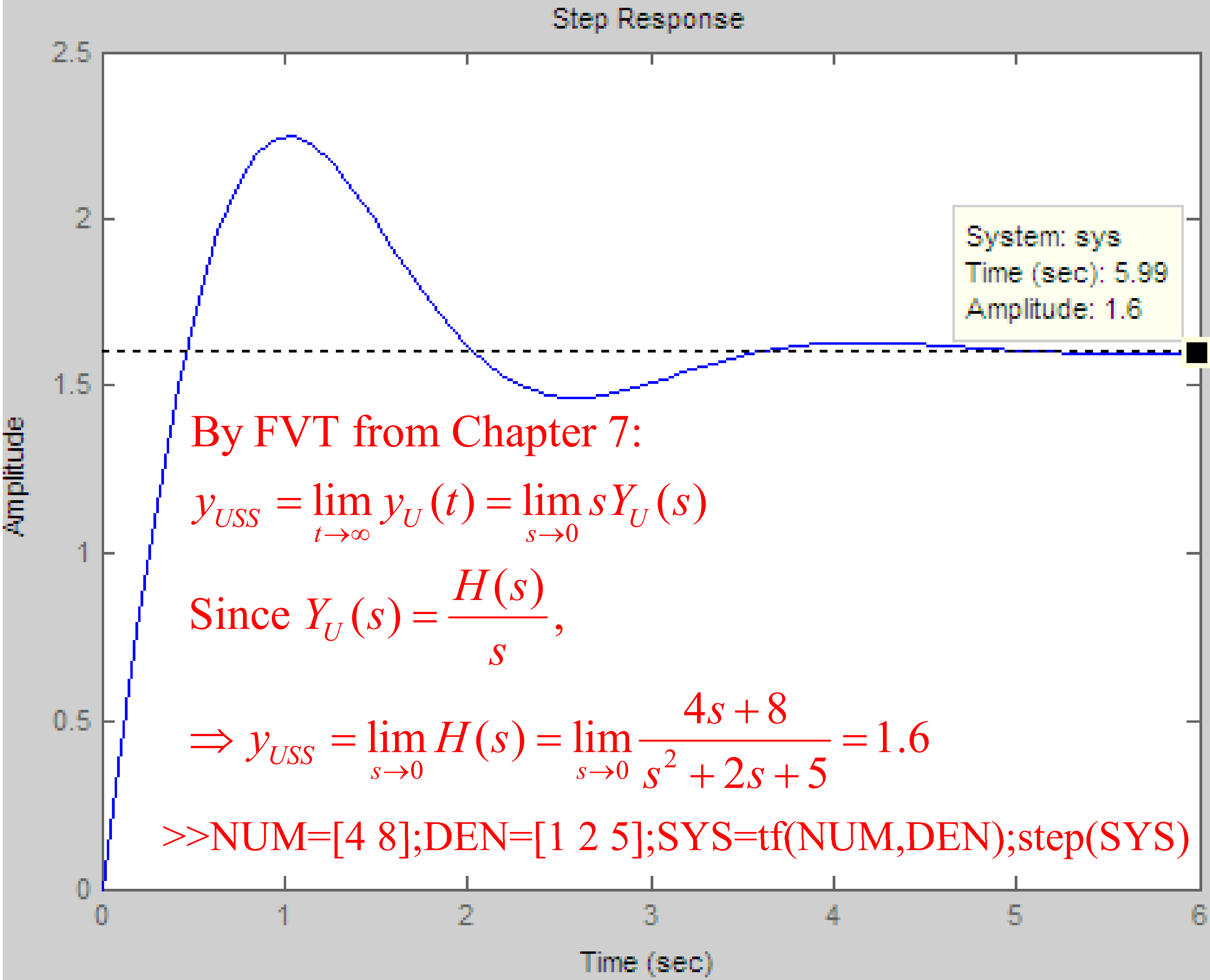
  >>NUM=[4 8];DEN=[1 2 5];SYS=tf(NUM,DEN);bode(SYS)

- Note, a Bode plot has a log scale on the frequency axis and a magnitude expressed in dB, i.e.,

$$X\ [dB] = 20\log_{10} X$$

$$\Rightarrow 1 \sim 0 \text{ dB}, \quad 0.707 \sim -3 \text{ dB}, \quad 1.414 \sim +3 \text{ dB}$$


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
10
0
-10
-20
-30
45
0
-45
-90
10^{-1}
10^{0}
10^{1}
10^{2}
The Magnitude of the Frequency Response at $\omega = 4$ [rad/s] is 2.3 [dB] ~ 1.3 as shown from TF
The Magnitude of the Frequency Response at $\omega = 0$ [rad/s] is 4.1 [dB] ~ $10^{(4.1/20)} = 1.6$ Compare this to the steady-state value of the system's unit step response (see next slide)
The Phase Angle of the Frequency Response at $\omega = 4$ [rad/s] is -80° or -1.406 [rad], as indicated by from TF



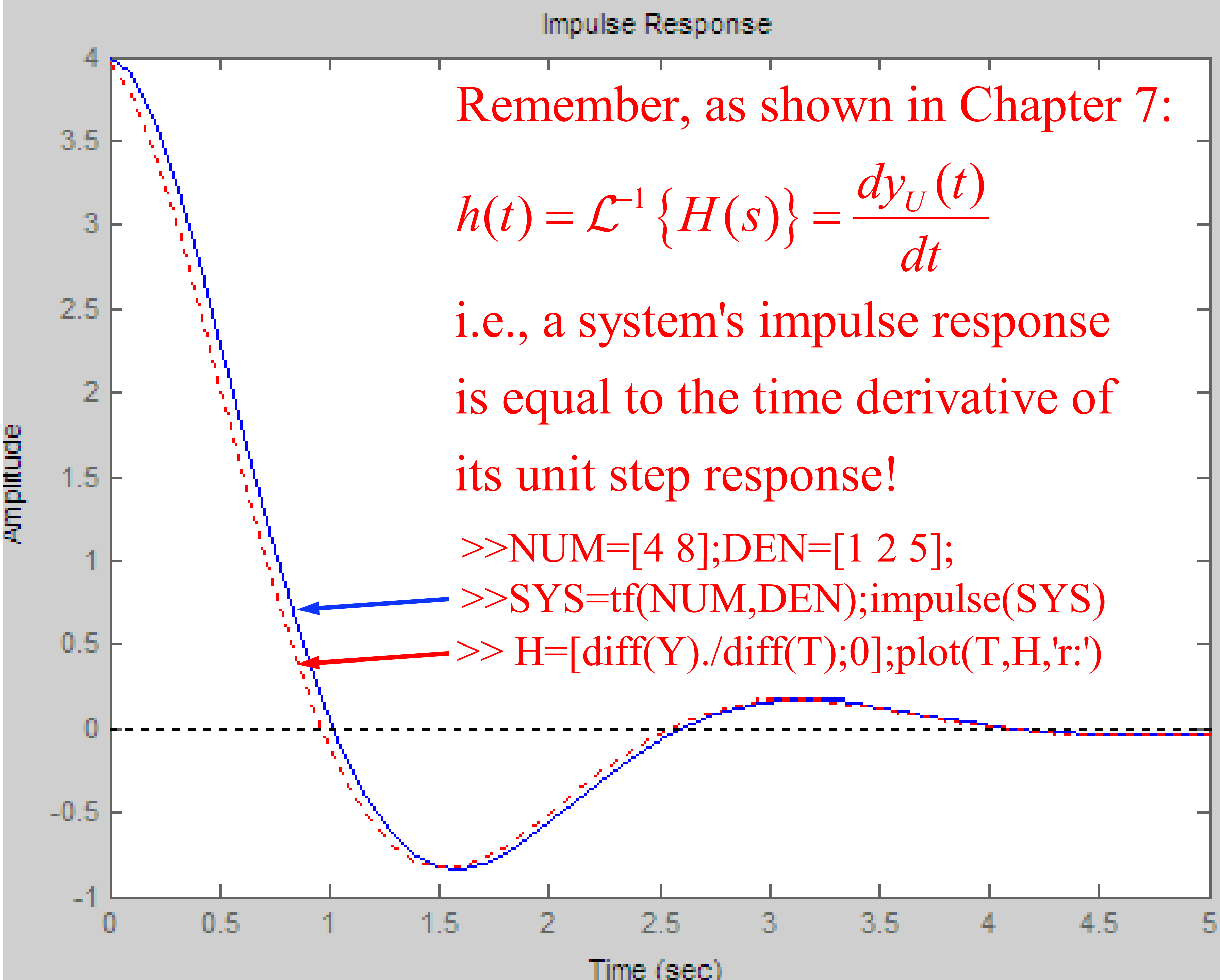
By FVT from Chapter 7:

$$y_{USS} = \lim_{t\to\infty} y_U(t) = \lim_{s\to 0} sY_U(s)$$

Since $Y_U(s) = \dfrac{H(s)}{s}$,

$$\Rightarrow y_{USS} = \lim_{s\to 0} H(s) = \lim_{s\to 0} \frac{4s+8}{s^2+2s+5} = 1.6$$

```
>>NUM=[4 8];DEN=[1 2 5];SYS=tf(NUM,DEN);step(SYS)
```



Remember, as shown in Chapter 7:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{dy_U(t)}{dt}$$

i.e., a system's impulse response is equal to the time derivative of its unit step response!

```
>>NUM=[4 8];DEN=[1 2 5];
>>SYS=tf(NUM,DEN);impulse(SYS)
>> H=[diff(Y)./diff(T);0];plot(T,H,'r:')
```

Fourier Series Example (Not Tested)

- Assume the prior system $H(s)$ is excited by a square wave of period T_0 defined for all integer p as:

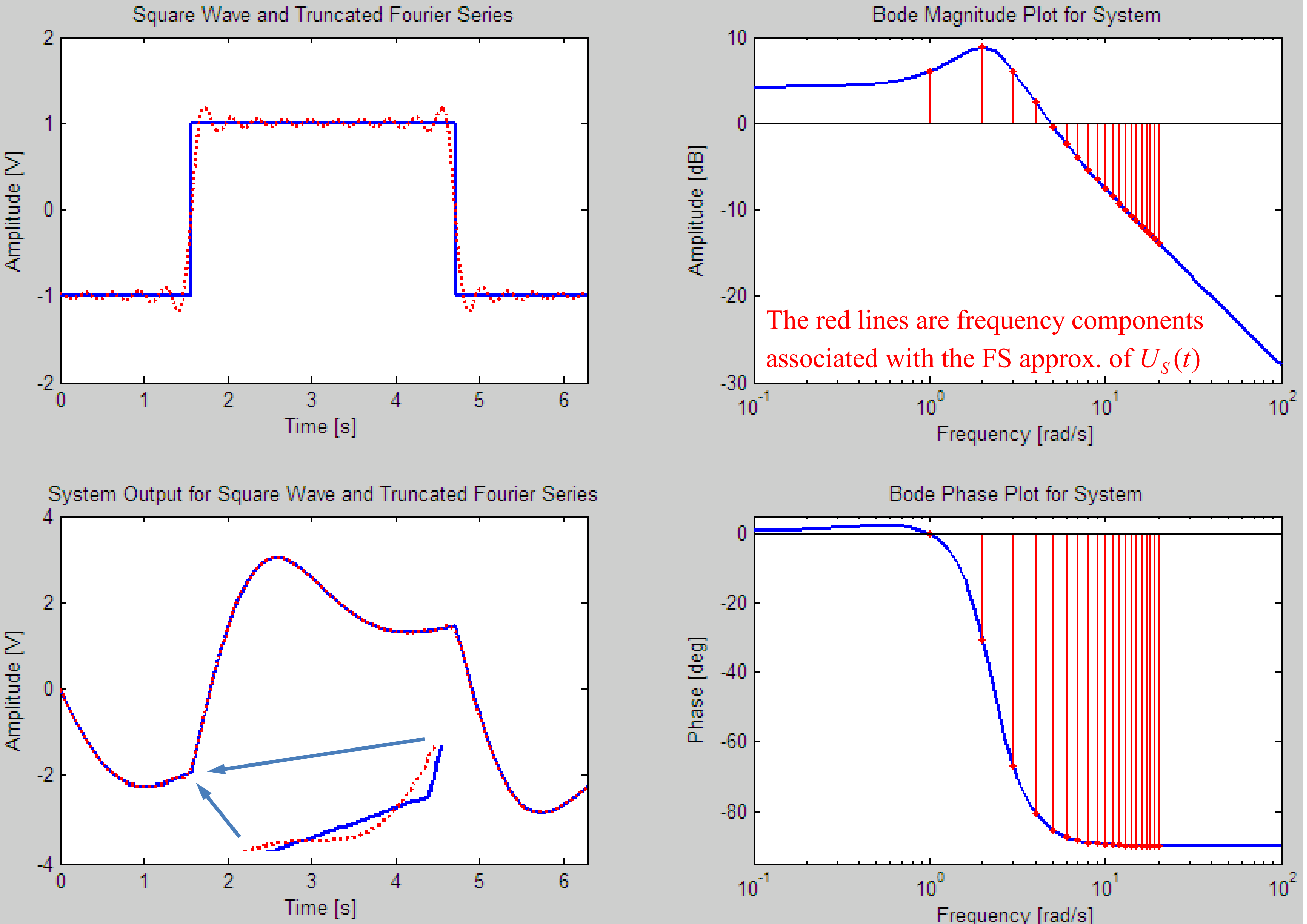
$$U_S(t) = \begin{cases} -1, \text{ for } \left(p - \frac{1}{4}\right)T_0 \le t < \left(p + \frac{1}{4}\right)T_0 \\ 1, \text{ for } \left(p + \frac{1}{4}\right)T_0 \le t < \left(p + \frac{3}{4}\right)T_0 \end{cases}$$

- The Fourier Series (FS) approx. of $U_S(t)$ is given by:

$$U_{SFS}(t) = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{-4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(n\omega_0 t\right) \text{ where } \omega_0 = 2\pi / T_0$$

- Approximate the response of the prior system to $U_S(t)$ using the first twenty terms of its FS (see following graphs)

Fourier Series Example (Not Tested)



The Impedance Concept

- Impedance is a frequency domain concept that can be expressed for various types of systems as the TF (or ratio) of flow to force

- For electrical systems, impedance is defined:

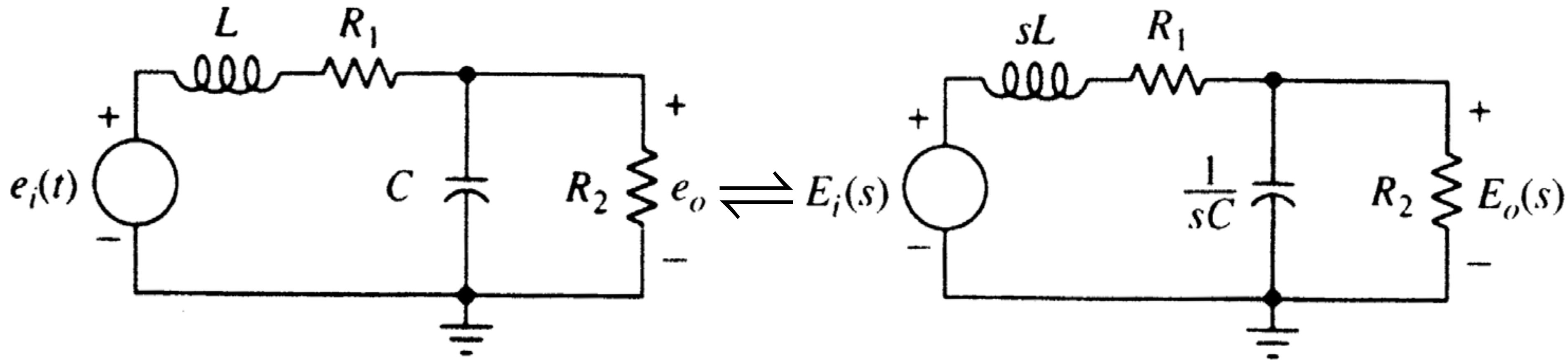
$$Z(s) \triangleq \frac{E(s)}{I(s)}$$

$$\Rightarrow Z_R(s) = R,\ Z_L(s) = Ls,\ Z_C(s) = \frac{1}{Cs}$$

- Impedances can be treated as a generalized resistances, even though they are functions of s

Electrical Impedance Ex.8.22: RC Circuit

- Replacing circuit elements with their equivalent impedances permits application of techniques used for pure resistive circuits, e.g., voltage-divider rule



- Combining parallel impedances at $E_o(s)$:

$$Z_2(s) = \frac{R_2/Cs}{R_2 + 1/Cs} = \frac{R_2}{1 + R_2 Cs}$$

Electrical Impedance Ex.8.22: RC Circuit

- Using voltage divider at $E_o(s)$:

$$E_o(s) = \left[\frac{Z_2(s)}{Ls + R_1 + Z_2(s)} \right] E_i(s)$$

$$= \left[\frac{\left({}^{1}\!/_{LC} \right)}{s^2 + \left({}^{L + R_1 R_2 C}\!/_{LCR_2} \right) s + \left({}^{R_1 + R_2}\!/_{LCR_2} \right)} \right] E_i(s)$$

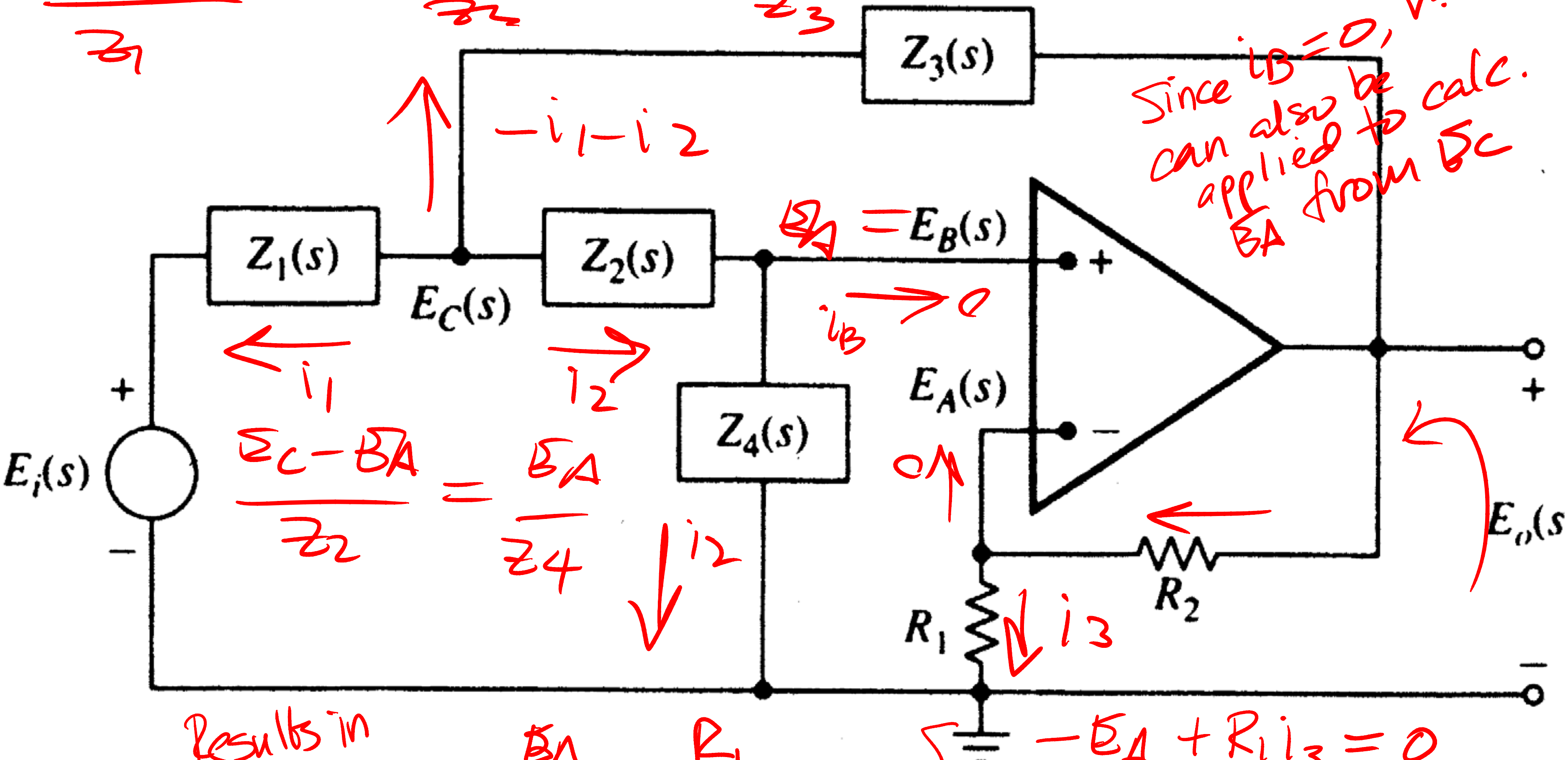
$$\rightleftharpoons \ddot{e}_o + \left({}^{L + R_1 R_2 C}\!/_{LCR_2} \right) \dot{e}_o + \left({}^{R_1 + R_2}\!/_{LCR_2} \right) e_o = \left({}^{1}\!/_{LC} \right) e_i(t)$$

$$\Rightarrow \text{If } e_i = AU(t),\ e_{oSS} = \left({}^{R_2}\!/_{R_1 + R_2} \right) A$$

Electrical Impedance Ex.8.24: Op-Amp Circuit

- Find the TF for the op-amp circuit, $H(s) = \dfrac{E_o(s)}{E_i(s)}$

$$\frac{E_C - E_i}{Z_1} + \frac{E_C - E_A}{Z_2} + \frac{E_C - E_o}{Z_3} = 0$$



Since $i_B = 0$, V÷ can also be applied to calc. E_A from E_C

$$\frac{E_C - E_A}{Z_2} = \frac{E_A}{Z_4}$$

Results in Voltage ÷:

$$\frac{E_A}{E_o} = \frac{R_1}{R_1 + R_2}$$

$$\begin{cases} -E_A + R_1 i_3 = 0 \\ -E_o + (R_2 + R_1) i_3 = 0 \end{cases}$$

Electrical Impedance Ex.8.24: Op-Amp Circuit

- Letting $K \triangleq \dfrac{R_1 + R_2}{R_1}$, we can repeatedly apply the voltage-divider rule to obtain:

$$E_A(s) = \frac{R_1}{R_1 + R_2} E_o(s) = \frac{1}{K} E_o(s)$$

$$E_B(s) = \frac{Z_4(s)}{Z_2(s) + Z_4(s)} E_C(s)$$

Due to the op-amp's virtual short, $E_A(s) = E_B(s)$

$$\Rightarrow E_C(s) = \frac{Z_2(s) + Z_4(s)}{K Z_4(s)} E_o(s)$$

Electrical Impedance Ex.8.24: Op-Amp Circuit

- KCL at Node C yields:

$$\frac{1}{Z_1(s)}\left[E_C(s)-E_i(s)\right]+\frac{1}{Z_3(s)}\left[E_C(s)-E_o(s)\right]$$

$$+\left[\frac{1}{Z_2(s)+Z_4(s)}\right]E_C(s)=0$$

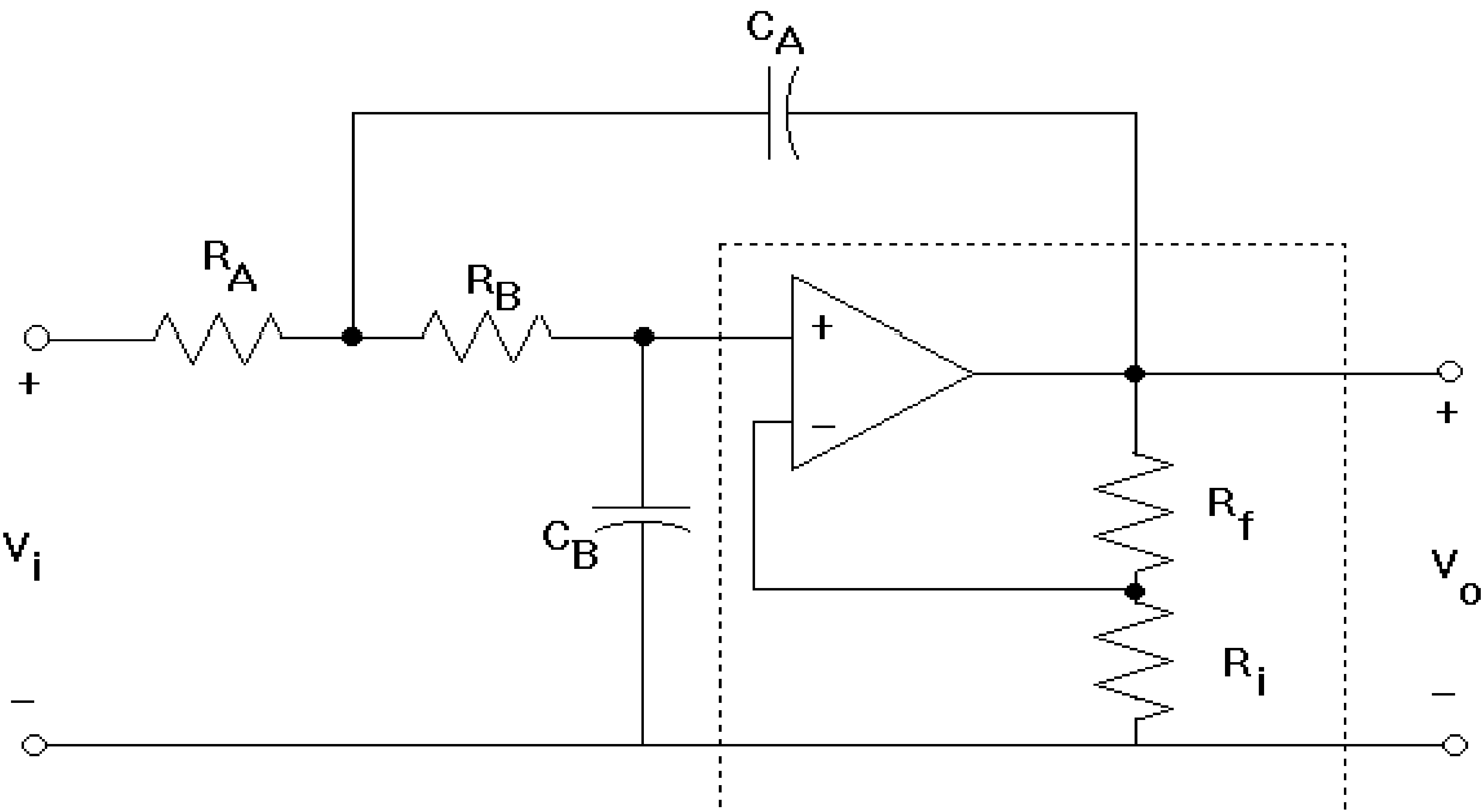
- Eliminating $E_C(s)$ and simplifying yields $H(s)\triangleq\dfrac{E_o(s)}{E_i(s)}$

$$=\frac{KZ_3(s)Z_4(s)}{Z_1(s)Z_2(s)+\left(1-K\right)Z_1(s)Z_4(s)+Z_3(s)\left[Z_1(s)+Z_2(s)+Z_4(s)\right]}$$

Sallen-Key Low-pass Filter

When $Z_1(s) = R_A,\ Z_2(s) = R_B,\ Z_3(s) = C_A,\ Z_4(s) = C_B,$

the circuit is referred to as a Sallen-Key Low-pass filter



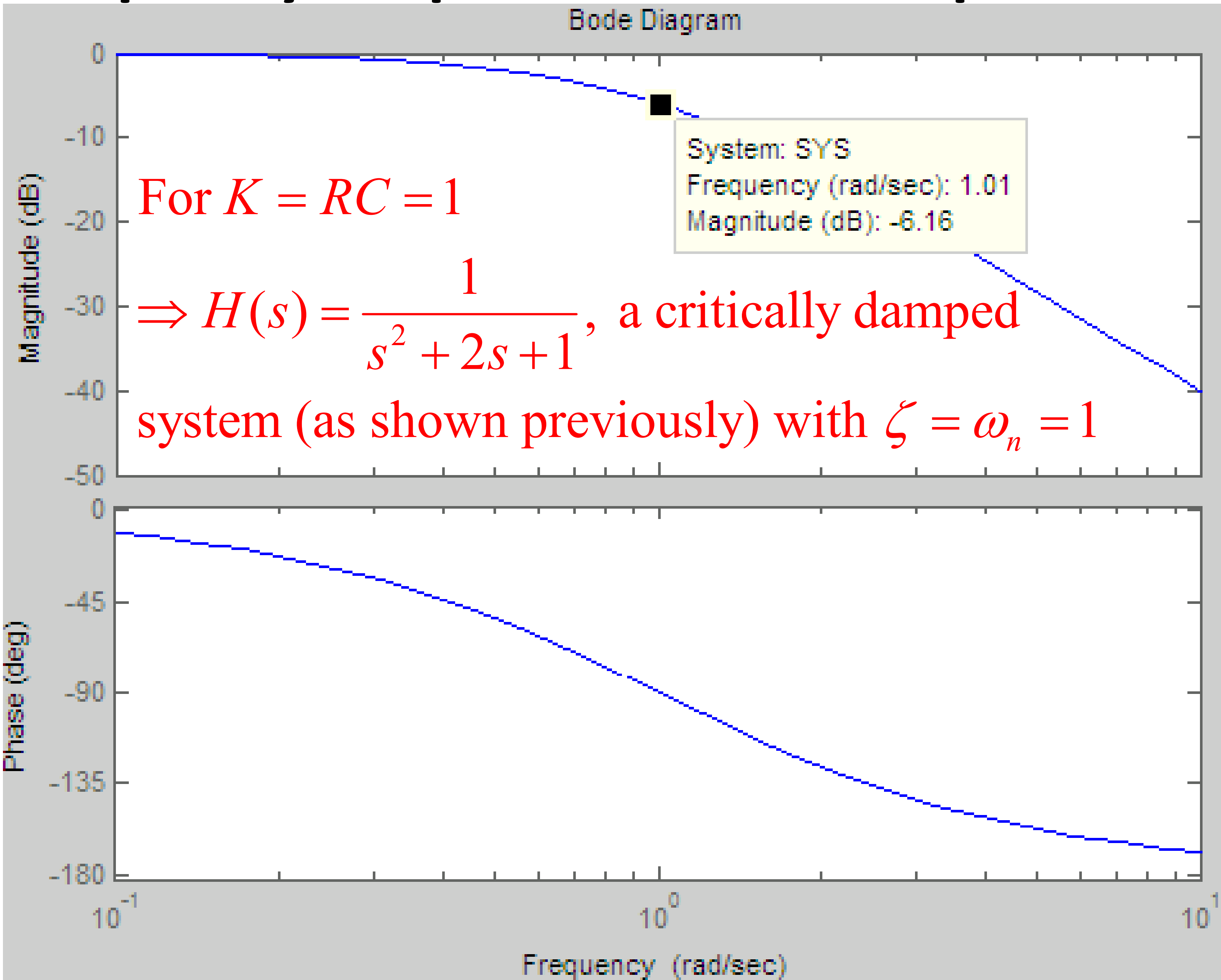
Sallen-Key Low-pass Filters

If $R_A = R_B = R,\ C_A = C_B = C,$ with $K = \left(\dfrac{R_f + R_i}{R_i} \right)$

$$H(s) \triangleq \frac{V_o(s)}{V_i(s)} = \frac{\dfrac{K}{(RC)^2}}{s^2 + \left(\dfrac{3-K}{RC} \right) s + \left(\dfrac{1}{RC} \right)^2};$$

the natural frequnecy of this low-pass filter can be tuned by selecting RC and damping by adjusting K

Frequency Response of S-K Low-pass Filter



For $K = RC = 1$

$$\Rightarrow H(s) = \frac{1}{s^2 + 2s + 1},$$ a critically damped system (as shown previously) with $\zeta = \omega_n = 1$

Mechanical Impedance

- For mechanical systems, impedance is defined:

$$Z(s) \triangleq \frac{F(s)}{v(s)}$$

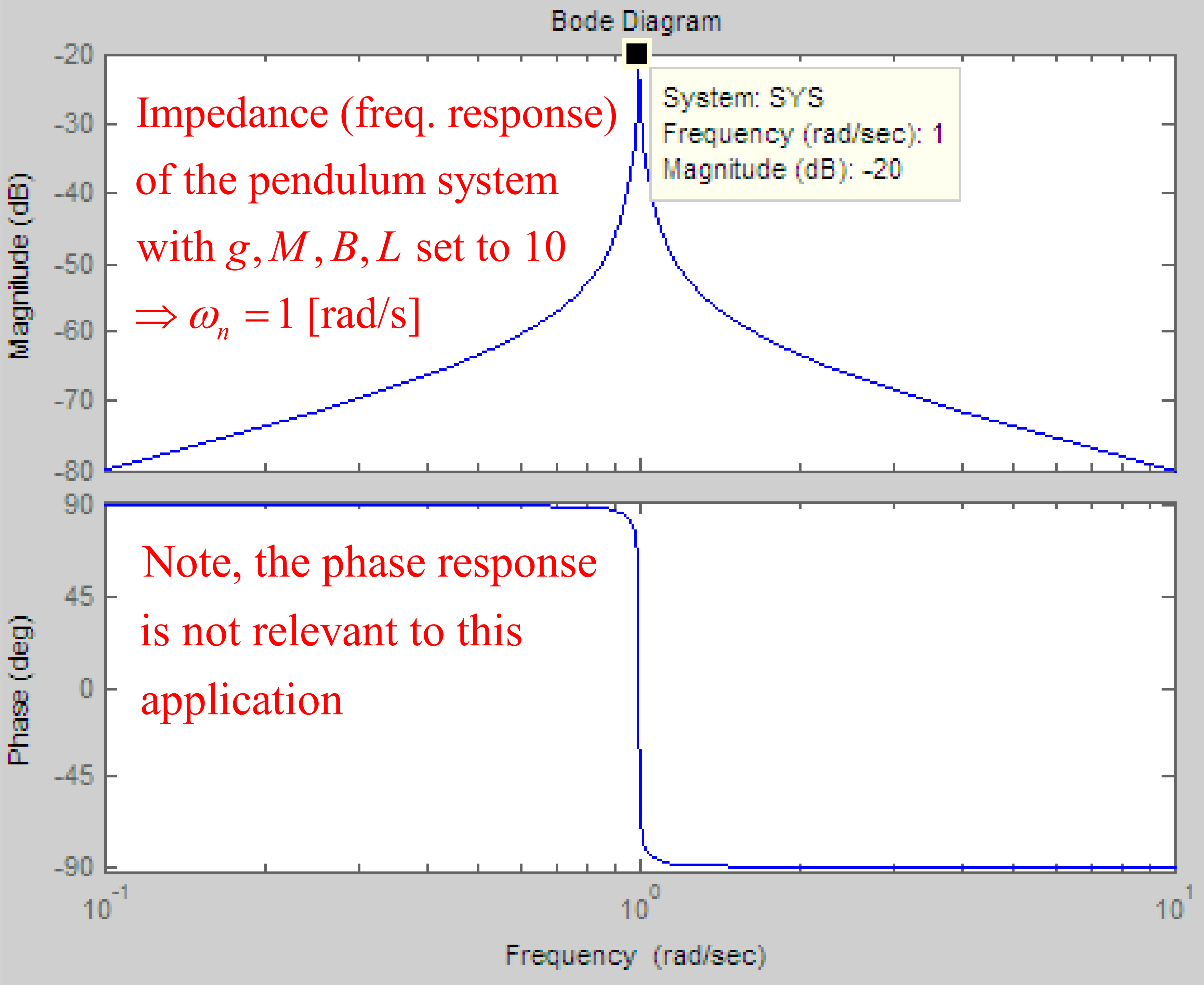
$$\Rightarrow Z_B(s) = B,\ Z_M(s) = Ms,\ Z_K(s) = \frac{K}{s}$$

- The mechanical impedance is a function of the frequency of the applied force and can vary greatly over frequency, e.g., for a pendulum (see pg. 116):

$$Z(s) \triangleq \frac{F(s)}{v(s)} = \frac{\left(1/ML^2\right)s}{s^2 + \left(B/ML^2\right)s + \left(g/L\right)} = \frac{G_{DC}\omega_n{}^2 s}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

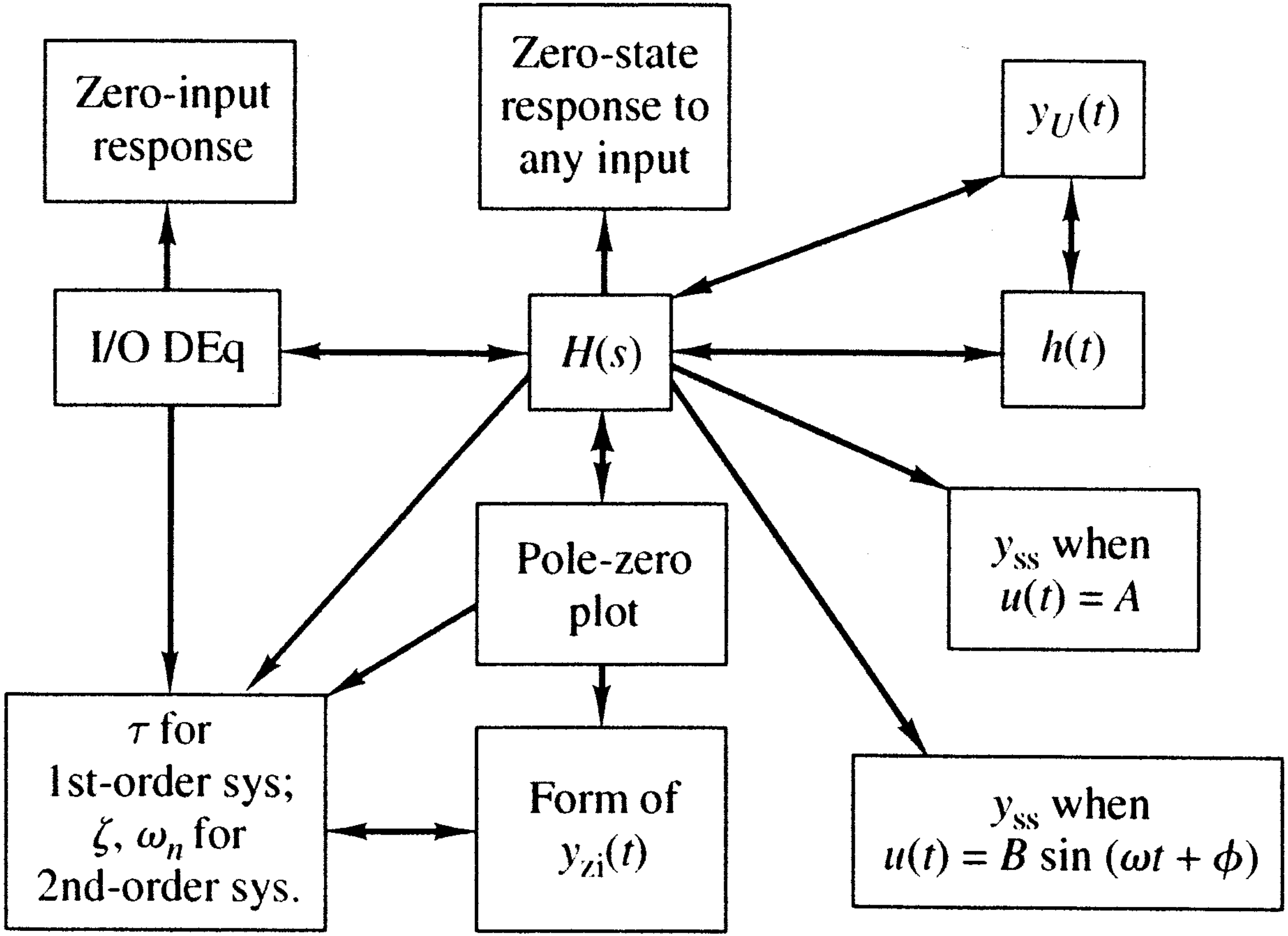
Mechanical Impedance

- At resonant (natural) frequencies, the mechanical impedance will be lower meaning less force/power is needed to cause a structure to move at a given velocity
- The simplest example of this is when a child pushes another on a swing; for the greatest swing amplitude the frequency of the pushes must be more-or-less at the resonant (or natural) frequency of the system
- Note, $\omega_n = \sqrt{g/L}$ for the simple pendulum system



Bode Diagram
System: SYS
Frequency (rad/sec): 1
Magnitude (dB): -20
Impedance (freq. response) of the pendulum system with g, M, B, L set to 10
$\Rightarrow \omega_n = 1$ [rad/s]
Note, the phase response is not relevant to this application
Magnitude (dB)
Phase (deg)
-20
-30
-40
-50
-60
-70
-80
90
45
0
-45
-90
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)

Questions?


Zero-input response
Zero-state response to any input
$y_U(t)$
I/O DEq
$H(s)$
$h(t)$
Pole-zero plot
y_{ss} when $u(t) = A$
τ for 1st-order sys; ζ, ω_n for 2nd-order sys.
Form of $y_{zi}(t)$
y_{ss} when $u(t) = B \sin(\omega t + \phi)$