Introduction

• This chapter introduces two more circuit elements, the capacitor and the inductor whose elements laws involve integration or differentiation; consequently, circuits containing capacitors and/or inductors:
  – are represented by differential equations, i.e., called dynamic circuits, as opposed to static circuits
  – able to store energy
  – have memory, meaning that the voltages and currents at a particular time depend past values
Introduction

• In addition, we will see that:
  – capacitor voltages and inductor currents are continuous functions of time assuming bounded currents/voltages, e.g., the voltage across a capacitor or the current through an inductor cannot change instantaneously
  – in a dc circuit, capacitors act like open circuits and inductors act like short circuits
  – a set of series or parallel capacitors can be reduced to an equivalent capacitor and a set of series or parallel inductors can be reduced to an equivalent inductor
  – an op amp and a capacitor can be used to make circuits that perform the mathematical operations of integration or differentiation
Capacitors

• A capacitor is a two-terminal element that can be modeled by two conducting plates separated by a nonconducting material, as shown below:

Capacitor law:

\[ q = Cv \]  \[ \Rightarrow \]  \[ i = C \frac{dv}{dt} \]

where

\[ C = \frac{\varepsilon A}{d} \]

Relative Dielectric Constant, \( \varepsilon_r \)

<table>
<thead>
<tr>
<th>MATERIAL</th>
<th>( \varepsilon_r )</th>
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<tbody>
<tr>
<td>Glass</td>
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<tr>
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<td>2</td>
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\( \varepsilon_0 = \text{permittivity of free space} = 8.85 \times 10^{12} \text{ farad/ meter} \)
Capacitors

- The circuit symbol for a capacitor is shown with the passive sign convention assumed:

\[ i = C \frac{dv}{dt} \]

**Capacitor law:**

\[ q = C v \]

where

\[ C = \varepsilon r \frac{A}{d} \]

*Relative Dielectric Constant, \( \varepsilon_r \)*

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\( \varepsilon_0 \) = permittivity of free space = \( 8.85 \times 10^{12} \) farad/meter.
Example 7.2-1: Capacitor Current & Voltage

- Find the current for a capacitor \( C = 1 \) mF when the voltage across the capacitor is represented by the signal below:

**Solution**

The voltage (with units of volts) is given by

\[
\begin{align*}
v &= 0 & t \leq 0 \\
&= 10t & 0 \leq t \leq 1 \\
&= 20 - 10t & 1 \leq t \leq 2 \\
&= 0 & t \geq 2
\end{align*}
\]
Example 7.2-1: Capacitor Current & Voltage

Then, since $i = C \frac{dv}{dt}$, where $C = 10^{-3}$ F, we obtain

$$i = \begin{cases} 
0 & t < 0 \\
10^{-2} & 0 < t < 1 \\
-10^{-2} & 1 < t < 2 \\
0 & t > 2 
\end{cases}$$
Capacitors

- We can solve for the voltage $v(t)$ across a capacitor in terms of the current $i(t)$ by integrating the element law:

  Capacitor law:

  \[ q = Cv \quad \Rightarrow \quad i = C \frac{dv}{dt} \]

  \[ \therefore \quad v = \frac{1}{C} \int_{-\infty}^{t} i \, d\tau = \frac{1}{C} \int_{t_0}^{t} i \, d\tau + v(t_0) \]

- The time $t_0$ is called the **initial time**, and the capacitor voltage $v(t_0)$ is called the **initial condition**; it is convenient to select $t_0 = 0$ as the initial time
Example 7.2-2: Capacitor Current & Voltage

• Find the voltage $v$ for a capacitor $C = 1/2 \ F$ when the current is represented by the signal below and $v(0) = 0$:

**Solution**

First, we write the equation for $i(t)$ as:

$$i = \begin{cases} 
0 & t \leq 0 \\
= t & 0 \leq t \leq 1 \\
= 1 & 1 \leq t \leq 2 \\
= 0 & 2 < t 
\end{cases}$$

It is important to note that the voltage waveform cannot change instantaneously, but the current waveform may do so.
Example 7.2-2: Capacitor Current & Voltage

Then since

\[ v = \frac{1}{C} \int_0^t i \, d\tau \]

and \( C = 1/2 \), we have \( v = 0 \) for \( t \leq 0 \)

\[ = 2 \int_0^t \tau \, d\tau \]

for \( 0 \leq t \leq 1 \)

\[ = 2 \int_1^t (1) \, d\tau + v(1) \]

for \( 1 \leq t \leq 2 \)

\[ = v(2) \]

for \( 2 \leq t \)

with units of volts. Therefore, for \( 0 < t \leq 1 \), we have

\[ v = t^2 \]

For the period \( 1 \leq t \leq 2 \), we note that \( v(1) = 1 \) and therefore we have

\[ v = 2(t - 1) + 1 = (2t - 1) \, V \]
Example 7.2-2: Capacitor Current & Voltage

- The resulting voltage waveform, as shown below, changes with $t^2$ during the first 1 s, linearly with $t$ during the period from 1 to 2 s, and stays constant equal to 3 V after $t = 2$ s:
Example 7.2-3 Capacitor Current & Voltage

• Determine the value of the capacitance associated with the current and voltage waveforms for the circuit below:
Example 7.2-3: Capacitor Current & Voltage

Solution

The current and voltage of the capacitor are related by

\[ v(t) - v(t_0) = \frac{1}{C} \int_{t_0}^{t} i(\tau) \, d\tau \]

Note, interpreting the equation graphically:

- \( v(t) - v(t_0) \) is the difference between the values of voltage at times \( t \) and \( t_0 \)
- \( \int_{t_0}^{t} i(\tau) \, d\tau \) is the area under the plot of \( i(t) \) versus \( t \), for times between \( t \) and \( t_0 \)

Pick convenient values \( t \) and \( t_0 \), for example, \( t_0 = 1 \) s and \( t = 3 \) s

\[ v(t) - v(t_0) = -1 - (-3) = 2 \text{ V} \]

and

\[ \int_{t_0}^{t} i(\tau) \, d\tau = \int_{1}^{3} 0.05 \, d\tau = (0.05)(3 - 1) = 0.1 \text{ A} \cdot \text{s} \]
Example 7.2-3 Capacitor Current & Voltage

Solution

\[ \therefore v(t) - v(t_0) = \frac{1}{C} \int_{t_0}^{t} i(\tau) \, d\tau \]

\[ \Rightarrow 2 = \frac{1}{C} (0.1) \quad \Rightarrow \quad C = 0.05 \frac{A \cdot s}{V} = 0.05 \text{F} = 50 \text{mF} \]
Example 7.2-4: Capacitor Current & Voltage

- Determine the values the constants, $a$ and $b$, used to label the plot of the capacitor current associated with the current and voltage waveforms for the circuit below:
Example 7.2-3: Capacitor Current & Voltage

**Solution**

The current and voltage of the capacitor are related by

\[
v(t) - v(t_0) = \frac{1}{C} \int_{t_0}^{t} i(\tau) \, d\tau
\]

Note, interpreting the equation graphically:

\[
24 = \left( \frac{1}{5 \mu} \right) a (5 - 2) \times 10^{-3} = 600a \Rightarrow a = 0.04
\]

\[
-24 = \left( \frac{1}{5 \mu} \right) b (7 - 5) \times 10^{-3} = 400b \Rightarrow b = -0.06
\]
Questions?
Energy Storage in a Capacitor

• The energy stored in a capacitor is:

\[ w_c(t) = \int_{-\infty}^{t} vi \, d\tau \]

• Since \( i = C \frac{dv}{dt} \) we have:

\[ w_c = \int_{-\infty}^{t} vC \frac{dv}{dt} \, d\tau = C \int_{v(-\infty)}^{v(t)} v \, dv = \frac{1}{2} Cv^2 \]

• Assuming the capacitor was uncharged at \( t = -\infty \), set \( v(-\infty) = 0 \); therefore:

\[ w_c(t) = \frac{1}{2} Cv^2(t) \, J \]
Energy Storage in a Capacitor

• For example, consider a 100-mF capacitor that has a voltage of 100 V across it. The energy stored is:

\[ w_c = \frac{1}{2} CV^2 = \frac{1}{2} (0.1)(100)^2 = 500 \text{ J} \]

• As long as the capacitor is not connected to any other element, the energy of 500 J remains stored.

• If we connect the capacitor to the terminals of a resistor, we expect a current to flow until all the energy is dissipated as heat by the resistor.
  – After all the energy dissipates, the current is zero and the voltage across the capacitor is zero.
Energy Storage in a Capacitor

• As noted in the previous section, the requirement of conservation of charge implies that the voltage on a capacitor is continuous
  – Thus, the *voltage and charge on a capacitor cannot change instantaneously*
  – This statement is summarized by the equation

\[ v(0^+) = v(0^-) \]
Energy Storage in a Capacitor

• For the circuit shown below assume the switch was closed for a long time and the capacitor voltage had become \( v_c = 10 \text{ V} \); then at time \( t = 0 \) we open the switch, as shown below.

• Since the voltage on the capacitor is continuous:

\[
\begin{align*}
v_c(0^+) &= v_c(0^-) = 10 \text{ V}
\end{align*}
\]
Ex 7.3-1: Energy Stored by a Capacitor

- A 10-mF capacitor is charged to 100 V, as shown in the circuit below; find the energy stored by the capacitor and the voltage of the capacitor at \( t = 0^+ \) after the switch is opened:

- **Solution** - The voltage of the capacitor is \( v = 100 \text{ V} \) at \( t = 0^- \); since the voltage at \( t = 0^+ \) is the same at \( t = 0^- \), we have:

\[
w_c = \frac{1}{2} Cv^2 = \frac{1}{2} (10^{-2})(100)^2 = 50 \text{ J}
\]
Ex 7.3-2: Power and Energy for a Capacitor

- The voltage across a 5-mF capacitor varies as shown below; determine and plot the capacitor current, power, and energy associated with the capacitor:
Solution

- The current is determined from \( i_c = C \frac{dv}{dt} \) as shown:
The power is $v(t)i(t)$, i.e., the product of the current curve and the voltage curve as shown below.

- Note, the capacitor receives energy during the first two seconds and then delivers energy for the period $2 < t < 3$: 
Solution

- The energy is $\omega = \int p \, dt$ and can be found as the area under the $p(t)$ curve, as shown below:
Questions?
Series and Parallel Capacitors

- The equivalent circuit for $N$ parallel capacitors:

$$C_p = C_1 + C_2 + C_3 + \cdots + C_N = \sum_{n=1}^{N} C_n$$
Series and Parallel Capacitors

• The equivalent circuit for $N$ series capacitors:

\[
\frac{1}{C_s} = \sum_{n=1}^{N} \frac{1}{C_n}
\]
Ex: 7.4-1 Parallel & Series Capacitors

- Find the equivalent capacitance for the circuit below when \( C_1 = C_2 = C_3 = 2 \text{ mF} \), \( v_1(0) = 10 \text{ V} \), and \( v_2(0) = v_3(0) = 20 \text{ V} \):

- Since \( C_2 \) and \( C_3 \) are in parallel, we replace them with \( C_p \), where:

\[
C_p = C_2 + C_3 = 4 \text{ mF}
\]
Ex: 7.4-1 Parallel & Series Capacitors

- Circuit resulting from replacing $C_2$ and $C_3$ with $C_p$:

- Next, replace the two series capacitors $C_1$ and $C_p$ with equivalent capacitor $C_s$ where:

\[
C_s = \frac{C_1C_p}{C_1 + C_p} = \frac{(2 \times 10^{-3})(4 \times 10^{-3})}{(2 \times 10^{-3}) + (4 \times 10^{-3})} = \frac{8}{6} \text{ mF}
\]
Ex: 7.4-1 Parallel & Series Capacitors

• To obtain the Equivalent circuit:

![Equivalent Circuit Diagram]

\[ C_s = \frac{8}{6} \text{ mF} \]

• Note, the initial voltage across the equivalent capacitor \( C_s \) can be computed as:

\[ v(0) = v_1(0) + v_p(0) = 10 + 20 = 30 \text{ V} \]
Exercise 7.4-1

- Find the equivalent capacitance of the circuit below:

\[
C_{eq} = \left( \left( 12 \text{mF} \parallel 4 \text{mF} \right) + 9 \text{mF} \right) \parallel 6 \text{mF} \\
= \left( 3 \text{mF} + 9 \text{mF} \right) \parallel 6 \text{mF} \\
= \left( 12 \text{mF} \right) \parallel 6 \text{mF} \\
= 4 \text{mF}
\]
Questions?
Inductors

• An inductor can be formed from a wire shaped as a multi-turn coil with the voltage across the inductor proportional to the rate of change of the current through the inductor, where $L$ is the constant of proportionality called \textit{inductance} measured in henrys (H), expressed as shown:

$$v = L \frac{di}{dt}$$

• Inductance is \textit{a measure of the ability of a device to store energy in the form of a magnetic field}
Inductors

• A magnetic flux $\phi(t)$ is associated with a current $i(t)$ in a coil.

• In the case of an $N$-turn coil where each line of flux passes through all coil turns; the total flux is given as $N\phi = iL$.

• Faraday found that a changing flux creates an induced voltage in each coil turn equal to the derivative of the flux $\phi$, so the total voltage $v$ across $N$ turns is:

$$v = N\frac{d\phi}{dt} = L\frac{di}{dt}$$

Note: Flux describes the strength and extent of an object's interaction with a magnetic field.
Inductors

• Inductors are wound in various forms with practical inductances ranging from 1 $\mu$H to 10 H.

• The circuit symbol for an inductor is shown with the passive sign convention assumed:

\[ v = L \frac{di}{dt} \]

• Conservation of flux applies to the inductor and therefore the flux, and thus current, through the inductor cannot have discontinuities, i.e., cannot change instantaneously.
Inductors

- We can solve for the current $i(t)$ across an inductor in terms of the voltage $v(t)$ by integrating the element law:

  Inductor law:

  $$v = L \frac{di}{dt}$$

  $\therefore$

  $$i = \frac{1}{L} \int_{t_0}^{t} v \, d\tau + i(t_0)$$

- The time $t_0$ is called the **initial time**, and the inductor current $i(t_0)$ is called the **initial condition**; it is convenient to select $t_0 = 0$ as the initial time.
Inductors

• Consider the voltage waveform shown below for an inductor when $L = 0.1 \text{ H}$ and $i(t_0) = 2 \text{ A}$:
Inductors

• Since \( v(t) = 2 \) V between \( t = 0 \) and \( t = 2 \), we have:

\[
i = 10 \int_{0}^{t} (2) \, d\tau + i(t_0) = 20t + 2 \text{ A}
\]

which results in the following current waveform:
Ex. 7.5-1: Inductor Current & Voltage

• Find the voltage across a $L = 0.1$ H inductor when the current in the inductor for $t > 0$ and $i(0) = 0$ is:

$$i = 20te^{-2t} \text{ A}$$
Solution

The voltage for $t > 0$ is given by:

$$v = L \frac{di}{dt} = (0.1) \frac{d}{dt}(20te^{-2t}) = 2\left[-2te^{-2t} + e^{-2t}\right] = 2e^{-2t}(1 - 2t) \text{ V}$$
Ex. 7.5-2: Inductor Current & Voltage

- The plots below represent the current and voltage of the inductor in the circuit; determine the value of L:
Solution

• The current and voltage of the inductor are related by:

\[ i(t) - i(t_0) = \frac{1}{L} \int_{t_0}^{t} \nu(\tau) \, d\tau \]

Note, interpreting the equation graphically:

\[ i(t) - i(t_0) = \text{the difference between the values of current at times } t \text{ and } t_0 \]

\[ \int_{t_0}^{t} \nu(\tau) \, d\tau = \text{the area under the plot of } \nu(t) \text{ versus } t \text{ for times between } t \text{ and } t_0 \]

• Picking convenient values for \( t \) and \( t_0 \), e.g., \( t_0 = 2 \text{ ms} \) and \( t = 6 \text{ ms} \) yields:

\[ i(t) - i(t_0) = 1 - (-2) = 3 \text{ A} \]

\[ \int_{t_0}^{t} \nu(\tau) \, d\tau = \int_{0.002}^{0.006} 30 \, d\tau = (30)(0.006 - 0.002) = 0.12 \text{ V} \cdot \text{s} \]
Solution

\[ i(t) - i(t_0) = \frac{1}{L} \int_{t_0}^{t} v(\tau) \, d\tau \]

\[ \Rightarrow 3 = \frac{1}{L} (0.12) \quad \Rightarrow \quad L = 0.040 \frac{V \cdot s}{A} = 0.040 \text{H} = 40 \text{mH} \]
Ex. 7.5-3: Inductor Current & Voltage

- The input voltage and current to the inductor circuit shown below are given by:
  \[ v(t) = 4e^{-20t} \text{V for } t > 0 \]
  \[ i(t) = -1.2e^{-20t} - 1.5 \text{A for } t > 0 \]

- Determine the values of the inductance, \( L \), and resistance, \( R \), if the initial inductor current is \( i_L(0) = -3.5 \text{ A} \)

Apply KCL at either node to get:

\[
i(t) = \frac{v(t)}{R} + i_L(t)
\]

\[
i(t) = \frac{v(t)}{R} + \left[ \frac{1}{L} \int_{0}^{t} v(\tau) d\tau + i(0) \right]
\]
Ex. 7.5-3: Inductor Current & Voltage

- Substituting in the time domain expressions for the specified inductor voltage and current yields:

\[
i(t) = \frac{v(t)}{R} + i_L(t) = \frac{v(t)}{R} + \left[ \frac{1}{L} \int_0^t v(\tau) d\tau + i(0) \right]
\]

\[
\Rightarrow -1.2e^{-20t} - 1.5 = \frac{4e^{-20t}}{R} + \frac{1}{L} \int_0^t 4e^{-20t} d\tau - 3.5
\]

\[
= \frac{4e^{-20t}}{R} + \frac{4}{L(-20)} \left(e^{-20t} - 1\right) - 3.5
\]

\[
= \left(\frac{4}{R} - \frac{1}{5L}\right)e^{-20t} + \frac{1}{5L} - 3.5
\]
Ex. 7.5-3: Inductor Current & Voltage

• Equating coefficients gives:

\[-1.5 = \frac{1}{5L} - 3.5 \Rightarrow L = 0.1 \text{H}\]

\[-1.2 = \frac{4}{R} - \frac{1}{5L} = \frac{4}{R} - \frac{1}{5(0.1)} = \frac{4}{R} - 2 \Rightarrow R = 5 \Omega\]
Questions?
Energy Storage in an Inductor

• Similar to a capacitor, the ideal inductor only stores energy; it is thus referred to as a passive element meaning it cannot generate or dissipate energy.

• The energy stored in an inductor can be calculated in a similar manner as we did for a capacitor; resulting in:

\[ w = \frac{1}{2}LI^2 \]
Ex. 7.6-1: Inductor Voltage & Current

• Find the current in a $L = 0.1\,\text{H}$ inductor when the initial current is zero and the voltage across the inductor is

$$v = 10te^{-5t}\,\text{V}$$

as shown below:
Ex. 7.6-1: Inductor Voltage & Current

• Given the voltage waveform, the current can be computed as:

\[
i = \frac{1}{L} \int_0^t v \, d\tau + i(t_0)
\]

\[
= 10 \int_0^t 10 \tau e^{-5\tau} \, d\tau = 100 \left[ \frac{-e^{-5t}}{25} (1 + 5\tau) \right]_0^t
\]

\[
= 4(1 - e^{-5t}(1 + 5t)) \text{ A}
\]
Ex. 7.6-2: Power & Energy for an Inductor

- Find the power and energy for an inductor of 0.1 H when the current and voltage are given as:

\[
\begin{align*}
  i &= 0 \quad t < 0 \\
  &= 20t \quad 0 \leq t \leq 1 \\
  &= 20 \quad 1 < t \\

  v &= 0 \quad t < 0 \\
  &= 2 \quad 0 < t < 1 \\
  &= 0 \quad 1 < t \\

  p &= vi = 40t \text{W} \\

  w &= \frac{1}{2}Li^2 \\
  &= 0.05(20t)^2 \quad 0 \leq t \leq 1 \\
  &= 0.05(20)^2 \quad 1 < t
\end{align*}
\]
Ex. 7.6-3: Power & Energy for an Inductor

- Find the power and the energy stored in a 0.1-H inductor when \( i = 20te^{-2t} \) A and \( v = 2e^{-2t}(1 - 2t) \) V for \( t \geq 0 \) and \( i = 0 \) for \( t < 0 \)

**Solution** - The power is given by:

\[
p = iv = (20te^{-2t})[2e^{-2t}(1 - 2t)] = 40te^{-4t}(1 - 2t) \text{ W} \quad t > 0
\]

- The energy is then given by:

\[
w = \frac{1}{2}LI^2 = 0.05(20te^{-2t})^2 = 20t^2e^{-4t} \text{ J} \quad t > 0
\]
Questions?
Series and Parallel Inductors

• A series connection of \( N \) inductors can be reduced to an equivalent series inductor \( L_s \), as shown:

\[
L_s = \sum_{n=1}^{N} L_n
\]
Series and Parallel Inductors

• A parallel connection of $N$ inductors can be reduced to an equivalent parallel inductor $L_p$, as shown:

\[
\frac{1}{L_p} = \sum_{n=1}^{N} \frac{1}{L_n}
\]
Ex. 7.7-1: Series & Parallel Inductors

• Find the equivalent inductance for the circuit of below assuming all the inductor currents are zero at $t_0$:

![Circuit Diagram]

• **Solution** - First, find the equivalent inductance for the 5-mH and 20-mH inductors in parallel, as shown:

$$\frac{1}{L_p} = \frac{1}{L_1} + \frac{1}{L_2} \implies L_p = \frac{L_1L_2}{L_1 + L_2} = \frac{5 \times 20}{5 + 20} = 4\text{mH}$$
Ex. 7.7-1: Series & Parallel Inductors

- This equivalent inductor is in series with the 2-mH and 3-mH inductors as shown below:

\[
\begin{align*}
L_{eq} &= \sum_{n=1}^{N} L_n = 2 + 3 + 4 = 9 \text{mH}
\end{align*}
\]

Therefore, we obtain:
Exercise 7.7-2

Find the equivalent inductance of the circuit shown below:

\[ L_{eq} = \left( (4\text{mH} \parallel 12\text{mH}) + 2\text{mH} \right) \parallel 20\text{mH} \]

\[ = \left( \left( \frac{4\text{m} \times 12\text{m}}{4\text{m}+12\text{m}} \right) + 2\text{mH} \right) \parallel 20\text{mH} \]

\[ = (3\text{mH} + 2\text{mH}) \parallel 20\text{mH} = 5\text{mH} \parallel 20\text{mH} \]

\[ = \frac{5\text{m} \times 20\text{m}}{5\text{m}+20\text{m}} = 4\text{mH} \]
Questions?
Initial Conditions of Switched Circuits

• In this section we concentrate on finding the change in selected variables in a circuit when a switch is thrown from open to closed or vice versa.

• The time of throwing the switch is considered to be $t = 0$, and we want to determine the value of the variable at $t = 0^-$ and $t = 0^+$, immediately before and after throwing the switch; thus, a switched circuit is a circuit with one or more switches that open or close at time $t_0 = 0$, as shown below:

By $I = I_L = I_s \frac{R_1}{(R_1 + R_2)}$
Initial Conditions of Switched Circuits

• We are particularly interested in the change in the current and voltage of energy storage elements after the switch is thrown since these variables along with the sources will dictate the behavior of the circuit for \( t > 0 \)

• We summarize the important characteristics of the behavior of an inductor and a capacitor on the next slide:
  – note that we assume \( t_0 = 0 \)
  – recall that an instantaneous change in the inductor current or the capacitor voltage is not permitted
  – however, it is possible to instantaneously change an inductor's voltage or a capacitor's current
### Characteristics of Energy Storage Elements

<table>
<thead>
<tr>
<th>VARIABLE</th>
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<th>CAPACITORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive sign convention</td>
<td><img src="image" alt="Inductor Diagram" /></td>
<td><img src="image" alt="Capacitor Diagram" /></td>
</tr>
<tr>
<td>Voltage</td>
<td>$v = L \frac{di}{dt}$</td>
<td>$v = \frac{1}{C} \int_0^t i , d\tau + v(0)$</td>
</tr>
<tr>
<td>Current</td>
<td>$i = \frac{1}{L} \int_0^t v , d\tau + i(0)$</td>
<td>$i = C \frac{dv}{dt}$</td>
</tr>
<tr>
<td>Power</td>
<td>$p = Li \frac{di}{dt}$</td>
<td>$p = C_v \frac{dv}{dt}$</td>
</tr>
<tr>
<td>Energy</td>
<td>$w = \frac{1}{2} Li^2$</td>
<td>$w = \frac{1}{2} C v^2$</td>
</tr>
</tbody>
</table>
Consider the switched inductor circuit shown previously:
- the switch is open at $t = 0^{-}$ and closes at $t = 0$
- the circuit has attained steady-state conditions before the switch is thrown (closed), so the inductor is a short circuit
- let $R_1 = R_2 = 1$, so that the source current at $t = 0^{-}$ divides equally between $i_1$ and $i_L$, with $i_L(0^-) = \left(\frac{1}{2}\right)^2 = 1A$

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<tr>
<td>An instantaneous change is not permitted for the element’s:</td>
<td>Current</td>
<td>Voltage</td>
</tr>
<tr>
<td>Will permit an instantaneous change in the element’s:</td>
<td>Voltage</td>
<td>Current</td>
</tr>
<tr>
<td>This element acts as a: (see note below)</td>
<td>Short circuit to a constant current into its terminals</td>
<td>Open circuit to a constant voltage across its terminals</td>
</tr>
</tbody>
</table>
Initial Conditions of Switched Circuits

- Since the current cannot change instantaneously for the inductor, we have \( i_L(0^+) = i_L(0^-) = 1 \text{A} \)

- However, the current in the resistor can change instantaneously; note prior to \( t = 0 \), we have \( i_1(0^-) = 1 \text{A} \)

- After the switch is thrown, we require that the voltage across \( R_1 \) be equal to zero because of the switched short circuit, and therefore \( i_1(0^+) = 0 \)

- Thus, the resistor current changes abruptly from 1 to 0 for \( t = 0^+ \)
Initial Conditions of Switched Circuits

• Now consider the switched capacitor circuit shown below:
• prior to $t = 0$, the switch has been closed for a long time
• since the source is a constant, the current in the capacitor is zero for $t < 0$ because the capacitor appears as an open circuit in a steady-state condition
• therefore, the voltage across the capacitor for $t < 0$ can be obtained using voltage divider as shown

$$v_c = \frac{R_2}{R_1 + R_2} v_s$$ for $t < 0$
Initial Conditions of Switched Circuits

• Therefore at $t = 0^-$ when $v_s = 10$ and $R_1 = R_2 = 1 \, \Omega$, we obtain:
  \[ v_c(0^-) = \left(\frac{1}{2}\right)10 = 5 \, \text{V} \]

• However, since the voltage across a capacitor cannot change instantaneously, we have
  \[ v_c(0^-) = v_c(0^+) = 5 \, \text{V} \]

• Thus, when the switch is opened at $t = 0$, the source is removed from the circuit, but $v_c(0^+)$ remains equal to 5 V
To analyze a circuit with both a capacitor and an inductor, as shown below, we must find $v_c(0^-)$ and $i_L(0^-)$ in steady-state:

By voltage divider:

$$v_c(0^-) = \left(\frac{3}{2 + 3}\right)10 = 6V = v_c(0^+)$$

$$i_L(0^-) = v_c(0^-)/3 = 2A = i_L(0^+)$$
Ex. 7.8-1: Initial Conditions in a Switched Circuit

- Find $i_L(0^+)$, $v_c(0^+)$, $dv_c(0^+)/dt$, and $di_L(0^+)/dt$ for the circuit below:
Solution

- Redraw the circuit for $t = 0^-$ by replacing the inductor with a short circuit and the capacitor with an open circuit, as shown below:

Note that:

- $i_L(0^-) = 0$
- $v_c(0^-) = -2\,\text{V}$
- $i_L(0^+) = 0$
- $v_c(0^+) = -2\,\text{V}$
Solution

• In order to find $dv_c(0^+)/dt$ and $di_L(0^+)/dt$, we throw the switches at $t = 0$ and redraw the circuit, as shown below:

• Since we wish to find $dv_c(0^+)/dt$, recall that

\[ i_c = C \frac{dv_c}{dt} \]

\[ \implies \frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C} \]
Solution

• To find $i_c$ we write KCL at node a to obtain

$$i_c + i_L + \frac{v_c - 10}{2} = 0$$

• Therefore, at $t = 0^+$

$$i_c(0^+) = \frac{10 - v_c(0^+)}{2} - i_L(0^+) = 6 - 0 = 6 \text{A}$$

• Hence, we obtain

$$\frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{6}{1/2} = 12 \text{V/s}$$
Solution

• Since we wish to find $\frac{di_L(0^+)}{dt}$, recall that

$$v_L = \frac{L}{dt} \frac{di_L}{dt} \quad \implies \quad \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

• To find $v_L$ we write KVL for the right-hand mesh to obtain

$$v_L - v_c + 1i_L = 0$$

• Therefore, at $t = 0^+$

$$v_L(0^+) = v_c(0^+) - i_L(0^+) = -2 - 0 = -2 \text{ V}$$

• Hence, we obtain

$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = -2 \text{ A/s}$$
Solution

• Thus, we find that at the switching time $t = 0$, the current in the inductor and the voltage of the capacitor remain constant.

• However, the inductor voltage changes instantaneously from $v_L(0^-) = 0$ to $v_L(0^+) = -2$ V, and we determined that $di_L(0^+)/dt = -2$ A/s.

• Also, the capacitor current changed instantaneously from $i_c(0^-) = 0$ to $i_c(0^+) = 6$ A and we found that $dv_c(0^+)/dt = 12$ V/s.
Questions?