ES250: Electrical Science

HW4: Node Voltage and Mesh Current Methods of Circuit Analysis

As of 2/14/10 at 11pm:
30% of students have yet to attempt HW#1
57% of students have yet to attempt HW#2
80% of students have yet to attempt HW#3
99% of students have yet to attempt HW#4
Node Voltage Equations

• Node voltage equations or simply “node equations” are a set of equations based on KCL that represent a circuit
  – unknown variables are the node voltages
  – after solving the node voltage equations, we determine the values of the element currents and voltages from the values of the node voltages
  – produces fewer eqns. in fewer variables then KCL

• It's easier to write node equations for certain types of circuit than for others; listed in order of ease:
  1. resistors and independent current sources
  2. resistors and independent current and voltage sources
  3. resistors and independent and dependent voltage and current sources
Node Voltage with Current Sources

- Circuit nodes are the places where elements connect together, e.g., nodes are labeled as node a, b, and c below:

- The node voltages are represented as $v_{ac}$ and $v_{bc}$ but it is conventional to drop the subscript $c$ and refer to these node voltages as simply $v_a$ and $v_b$
Node Voltage with Current Sources

- The reference node $v_{cc} = v_c = 0$ V, since a voltmeter measuring the node voltage at the reference node would have both probes connected to the same point.

Arbitrary reference node
Node Voltage Equations

- The unknown variables in node equations are the node voltages, determined by solving the node equations.
- To write a set of node equations, we do two things:
  1. express element currents as functions of the node voltages
  2. apply Kirchhoff’s current law (KCL) at each of the nodes of the circuit, except for the reference node.

KCL at node a gives:

\[ i_s = \frac{v_a}{R_2} + \frac{v_a - v_b}{R_1} \]

KCL at node b gives:

\[ \frac{v_a - v_b}{R_1} = \frac{v_b}{R_3} \]
Node Voltage Equations

KCL at node a gives:

\[
\frac{v_a - v_b}{R_1} = i_s + \frac{(v_a - v_b)}{R_2} + \frac{v_a - v_b}{R_3}
\]

KCL at node b gives:

\[
\frac{v_a - v_b}{R_1} = \frac{v_b}{R_3}
\]

• If \( R_1 = 1 \, \Omega \), \( R_2 = R_3 = 0.5 \, \Omega \), and \( i_s = 4 \, \text{A} \), solving the two eqns. in two unknowns yields:

\[
v_a = \frac{3}{2} \, \text{V} \quad \text{and} \quad v_b = \frac{1}{2} \, \text{V}
\]
Example 4.2-2: Node Equations

- Obtain the node equations for the circuit letting $v_a$ denote the node voltage at node a, $v_b$ denote the node voltage at node b, and $v_c$ denote the node voltage at node c with the reference node as shown.
Example 4.2-2: Node Equations

• Apply KCL at node a to obtain:

\[- \left( \frac{v_a - v_c}{R_1} \right) + i_1 - \left( \frac{v_a - v_c}{R_2} \right) + i_2 - \left( \frac{v_a - v_b}{R_5} \right) = 0\]

• Separate the terms of this equation that involve $v_a$ from the terms that involve $v_b$ and the terms that involve $v_c$ to obtain:

\[
\left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_5} \right)v_a - \left( \frac{1}{R_5} \right)v_b - \left( \frac{1}{R_1} + \frac{1}{R_2} \right)v_c = i_1 + i_2
\]

• Note the pattern in the node eqns. of circuits that contain only resistors and current sources:
  – in the node eqn. at node a, the coefficient of $v_a$ is the sum of the reciprocals of the resistances of all resistors connected to node a
  – the coefficients of $v_b$ (and likewise $v_c$) are minus the sum of the reciprocals of the resistances of resistors connected between nodes b and node a (or nodes c and node a for $v_c$
  – the right-hand side of this eqn. is the algebraic sum of current source currents directed into node a
Example 4.2-2: Node Equations

- Apply KCL at node b to obtain:
  \[-i_2 + \left( \frac{v_a - v_b}{R_5} \right) - \left( \frac{v_b - v_c}{R_3} \right) - \left( \frac{v_b}{R_4} \right) + i_3 = 0\]

- Separate the terms of this equation that involve \(v_b\) from the terms that involve \(v_a\) and the terms that involve \(v_c\) to obtain:
  \[- \left( \frac{1}{R_5} \right)v_a + \left( \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} \right)v_b - \left( \frac{1}{R_3} \right)v_c = i_3 - i_2\]

- Using the pattern, we write the KCL at node c to obtain:
  \[- \left( \frac{1}{R_1} + \frac{1}{R_2} \right)v_a - \left( \frac{1}{R_3} \right)v_b + \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_6} \right)v_c = -i_1\]

- Now determine the node voltages for the circuit when \(i_1 = 1\) A, \(i_2 = 2\) A, \(i_3 = 3\) A, \(R_1 = 5\) Ω, \(R_2 = 2\) Ω, \(R_3 = 10\) Ω, \(R_4 = 4\) Ω, \(R_5 = 5\) Ω, and \(R_6 = 2\) Ω using a matrix eqn. suitable for solution using MATLAB.
Solution

• The node equations are:

\[
\begin{align*}
\left(\frac{1}{5} + \frac{1}{2} + \frac{1}{5}\right)\nu_a - \left(\frac{1}{5}\right)\nu_b - \left(\frac{1}{5} + \frac{1}{2}\right)\nu_c &= 1 + 2 \\
-\left(\frac{1}{5}\right)\nu_a + \left(\frac{1}{10} + \frac{1}{5} + \frac{1}{4}\right)\nu_b - \left(\frac{1}{10}\right)\nu_c &= -2 + 3 \\
-\left(\frac{1}{5} + \frac{1}{2}\right)\nu_a - \left(\frac{1}{10}\right)\nu_b + \left(\frac{1}{5} + \frac{1}{2} + \frac{1}{10} + \frac{1}{2}\right)\nu_c &= -1
\end{align*}
\]

or:

\[
\begin{align*}
0.9\nu_a - 0.2\nu_b - 0.7\nu_c &= 3 \\
-0.2\nu_a + 0.55\nu_b - 0.1\nu_c &= 1 \\
-0.7\nu_a - 0.1\nu_b + 1.3\nu_c &= -1
\end{align*}
\]

• The node equations can be written using matrices as

\[
\begin{bmatrix}
0.9 & -0.2 & -0.7 \\
-0.2 & 0.55 & -0.1 \\
-0.7 & -0.1 & 1.3
\end{bmatrix}
\begin{bmatrix}
\nu_a \\
\nu_b \\
\nu_c
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
1 \\
-1
\end{bmatrix}
\sim
G\nu = I \Rightarrow \nu = G^{-1}I
\]

or \( G \backslash I \) in MATLAB
MATLAB Solution

>> G=[0.9 -0.2 -0.7; -0.2 0.55 -0.1; -0.7 -0.1 1.3]
G =
    0.9000   -0.2000   -0.7000
   -0.2000    0.5500   -0.1000
   -0.7000   -0.1000    1.3000

>> l=[3; 1; -1]
l =
    3
    1
   -1

>> v=G\l
v =
    7.1579
    5.0526
    3.4737
Node Voltage Analysis of Circuits with Current and Voltage Sources

- Consider the circuit with an independent voltage source and an independent current source, as shown:

Thus, \( v_a = v_s \) is known and only \( v_b \) is unknown; so we write a KCL equation at node b to obtain:

\[
 i_s = \frac{v_b}{R_3} + \frac{v_b - v_a}{R_2} = \frac{v_b}{R_3} + \frac{v_b - v_s}{R_2} \quad \Rightarrow \quad v_b = \frac{R_2 R_3 i_s + R_3 v_s}{R_2 + R_3}
\]
Circuits with a Supernode

• Consider the circuit with an independent voltage source and an independent current source, as shown:

By KVL:
\[ v_a = v_s + v_b \]

By KCL:
\[ \frac{v_a}{R_1} + \frac{v_b}{R_2} = i_s \]

⇒ \[ \frac{v_s + v_b}{R_1} + \frac{v_b}{R_2} = i_s \]

⇒ \[ v_b = \frac{R_1 R_2 i_s - R_2 v_s}{R_1 + R_2} \]

• Since \( v_a \) and \( v_b \) are dependent, we consider nodes a and b as part of one larger supernode represented by the shaded ellipse; by KCL the algebraic sum of the currents entering a supernode is zero which means that we apply KCL to a supernode in the same way as a regular node.
Example 4.3-2: Supernodes

• Consider the circuit with an independent voltage source and an independent current sources, as shown:

• Apply KCL and KVL to the supernode to get:

\[ 1.5 = \frac{v_a}{6} + 3.5 + \frac{v_b}{3} \Rightarrow \frac{v_a}{6} + \frac{v_b}{3} = -2.0 \quad \text{and} \quad v_b - v_a = 12 \]

\[ \Rightarrow v_a = -12 \text{ V} \quad \text{and} \quad v_b = 0 \text{ V} \]

Note that \( v_b \) is 0 V, i.e., the same as the ground reference, yet 3.5A flows through a grounded branch.
Example 4.3-3 Node Equations for a Circuit Containing Voltage Sources

• Determine the node voltages for the circuit shown:

By KVL:
\[ v_b = -12 \text{ V} \]
\[ v_a = v_c + 10 \]

By KCL at supernode:
\[ \frac{v_a - v_b}{10} + 2 + \frac{v_c - v_b}{40} = 5 \]
\[ \Rightarrow 4v_a + v_c - 5v_b = 120 \]

• Solving these eqns. for \( v_c \) we get:
\[ v_c = 4 \text{ V} \]
Node Voltage Analysis with Dependent Sources

- When a circuit contains dependent sources, the controlling current or voltage of the dependent sources must be expressed as functions of the node voltages, e.g.:

  ![Circuit Diagram]

  **By Ohm’s Law:**
  \[ i_x = \frac{v_a - v_b}{6} \]

  **By KVL:**
  \[ v_a = 8 \text{ V} \]
  \[ v_c = 3i_x = 3\left(\frac{8 - v_b}{6}\right) = 4 - \frac{v_b}{2} \]

  **By KCL @ node b:**
  \[ \frac{8 - v_b}{6} + 2 = \frac{v_b - v_c}{3} \]
  \[ \Rightarrow \left\{ \begin{array}{l}
  \text{solve for 2 eqns. in 2 unknowns} \\
  v_b = 7 \text{ V} \\
  v_c = \frac{1}{2} \text{ V}
  \end{array} \right. \]
Example 4.4-2: Circuit with a VCVS

- Determine the node voltages for the circuit shown:

First express control voltage $v_x$ as a fcn. of the node voltages using KVL:

By KVL:

$\begin{align*}
\nu_a - \nu_b &= 4 \nu_x \\
&= 4(-\nu_a) = -4\nu_a \\
\Rightarrow \quad \nu_b &= 5\nu_a
\end{align*}$

By KCL@ the supernode:

$3 = \frac{\nu_a}{4} + \frac{\nu_b}{10}$

$\Rightarrow \begin{cases} 
\text{solve for 2 eqns.} \\
\text{in 2 unknowns} \\
\nu_a &= 4\text{V} \\
\nu_b &= 5\nu_a = 20\text{V}
\end{cases}$
Questions?
Mesh Current Equations

- Mesh current equations or simply “mesh equations” are a set of equations based on KVL that represent a circuit
  - unknown variables are the mesh currents
  - after solving the mesh current equations, we determine the values of the element currents and voltages from the values of the mesh currents
  - produces fewer eqns. in fewer variables then KCL
- It's easier to write mesh current equations for some types of circuit than for others; listed in order of ease:
  1. resistors and independent voltage sources
  2. resistors and independent current and voltage sources
  3. resistors and independent and dependent voltage and current sources
Mesh Current w/Independent Voltage Sources

- A *mesh* loop that does not contain any other loops within it
  - mesh current analysis is applicable only to planar circuits, i.e., that can be drawn on a plane without crossovers as illustrated below:

```
Note: the wires cross not are not connected
```
Mesh Current w/Independent Voltage Sources

- For planar networks the meshes look like “windows”
- there are four meshes in the circuit below identified as \( M_i \)
- mesh 2 contains the elements \( R_3, R_4, \) and \( R_5 \), with \( R_3 \) common to both mesh 1 and mesh 2
Mesh Current w/Independent Voltage Sources

- A circuit with two mesh currents $i_1$ and $i_2$ is shown on the left-hand side below:
  - we will assume mesh currents flow clockwise
  - the figure on the right shows how ammeters could be inserted in the circuit to measure the mesh currents
Mesh Current w/Independent Voltage Sources

• To write a set of mesh equations, we do two things:
  1. express element voltages as functions of the mesh currents
  2. apply KVL to each of the circuit meshes

• The current of element B has been labeled as $i_b$; applying KCL at node b gives:

$$i_b = i_1 - i_2$$

or

$$i'_b = i_2 - i_1 = -i_b$$
Mesh Current w/Independent Voltage Sources

- Write mesh equations to represent the circuit below using the standard KVL convention, i.e., add voltages when the + reference polarity of an element voltage is encountered before the – sign, and vice versa:

By KVL @ mesh 1:

\[-v_s + R_1 i_1 + R_3 (i_1 - i_2) = 0\]

By KVL @ mesh 2:

\[-R_3 (i_1 - i_2) + R_2 i_2 = 0\]
Mesh Current w/Independent Voltage Sources

• Solve the circuit with three mesh currents and two voltage sources:

mesh 1: \[-v_s + R_1i_1 + R_4(i_1 - i_2) = 0\]
mesh 2: \[R_2i_2 + R_5(i_2 - i_3) + R_4(i_2 - i_1) = 0\]
mesh 3: \[R_5(i_3 - i_2) + R_3i_3 + v_g = 0\]
Mesh Current w/Independent Voltage Sources

• These three mesh equations can be rewritten by collecting coefficients for each mesh current as:

mesh 1: \[(R_1 + R_4)i_1 - R_4i_2 = v_s\]

mesh 2: \[-R_4i_1 + (R_4 + R_2 + R_5)i_2 - R_5i_3 = 0\]

mesh 3: \[-R_5i_2 + (R_3 + R_5)i_3 = -v_g\]

• The matrix eqn. for solving the mesh analysis: \[\mathbf{Ri} = v_s\]

• Note that \(\mathbf{R}\) is a symmetric matrix, as shown:

\[
\mathbf{R} = \begin{bmatrix}
(R_1 + R_4) & -R_4 & 0 \\
-R_4 & (R_2 + R_4 + R_5) & -R_5 \\
0 & -R_5 & (R_3 + R_5)
\end{bmatrix}
\]

• This eqn. can be solved using MATLAB as shown: \[\mathbf{i} = \mathbf{R} \backslash \mathbf{v}\]
Mesh Current w/Independent Current Sources

• Circuit with an independent voltage source and an independent current source:

Due to the current source, \( i_2 = -i_s \), is known; thus we only need to determine the first mesh current \( i_1 \), as shown:

\[
(R_1 + R_2)i_1 - R_2i_2 = v_s \quad \Rightarrow \quad i_1 = \frac{v_s - R_2i_s}{R_1 + R_2}
\]

• This eqn. can be solved using MATLAB as shown:
Circuits with a Supermesh

A *supermesh* is one mesh created from two meshes with a common current source, e.g., the supermesh below incorporates meshes 1 and 2 as shown by the dashed line:

\[
\begin{align*}
\text{supermesh:} & \quad 1i_1 + 5i_2 - 4i_3 = 10 \\
\text{mesh 3:} & \quad -1i_1 - 3i_2 + 6i_3 = 0 \\
\text{current source:} & \quad i_1 - i_2 = 5
\end{align*}
\]
Example 4.6-2: A Supermesh Circuit

- Apply KVL to the supermesh circuit below to obtain:

\[
\text{supermesh: } 9i_1 + 3i_2 + 6i_2 - 12 = 0 \Rightarrow 9i_1 + 9i_2 = 12
\]

- current source: \( i_1 = i_2 + 1.5 \)

- Solving the simultaneous equations yields:

\[
i_1 = 1.4167 \text{ A} \quad \text{and} \quad i_2 = -83.3 \text{ mA}
\]
Mesh Current Analysis with Dependent Sources

• When a circuit contains dependent sources, the controlling current or voltage of the dependent sources must be expressed as a function of the mesh currents
  – it is then a simple matter to express the controlled current or voltage as a function of the mesh currents
  – the mesh equations are then obtained using KVL

• The controlling current of the dependent source, $i_a$, is the current in a short circuit: $i_a = i_1 - i_2$
Mesh Current Analysis with Dependent Sources

The dependent source is in only mesh 2, hence: $5i_a = -i_2$

Solving for $i_2$ gives: $i_2 = -5i_a = -5(i_1 - i_2)$

$\Rightarrow -4i_2 = -5i_1 \Rightarrow i_2 = \frac{5}{4}i_1$

Apply KVL to mesh 1 to get: $32i_1 - 24 = 0 \Rightarrow i_1 = \frac{3}{4} \text{ A}$

$\Rightarrow i_2 = \frac{5}{4} \left(\frac{3}{4}\right) = \frac{15}{16} \text{ A}$

Apply KVL to mesh 2:
$32i_2 - v_m = 0 \Rightarrow v_m = 32i_2 = 32 \left(\frac{15}{16}\right) = 30 \text{ V}$
Questions?