ES250:
Electrical Science

HW3: Voltage and Current Division
Series Resistors and Voltage Division

- A *series* connection of elements carries a common current, i.e., two elements connected to a single node, e.g.:
  - $R_1$ and $R_2$ are the only elements connected to node b; consequently, $i_1 = i_2$, and the resistors are in series
  - since all resistors are in series the voltages and currents will not change if we interchange the resistor positions
Series Resistors and Voltage Division

• The prior circuit demonstrates the principle of voltage division with the circuit is called a voltage divider.

• In general, we may represent the voltage divider principle by the equation:

\[ v_n = \frac{R_n}{R_1 + R_2 + \ldots + R_N} v_s \]

• For example, the voltage across resistor \( R_2 \) is given by:

\[ v_2 = \frac{R_2}{R_1 + R_2 + R_3} v_s \]
Series Resistors and Voltage Division

• In general, the series connection of \( N \) resistors having resistances \( R_1, R_2 \ldots R_N \) is equivalent to the single resistor having resistance:

\[
R_s = R_1 + R_2 + \ldots + R_N
\]

• Replacing series resistors by an equivalent resistor does not change the current or voltage of any other circuit element:

\[
p = i_s^2(R_1 + R_2 + R_3) = i_s^2R_s
\]
Example 3.3-1: Voltage Divider

• Find $R_2$ so that $v_2$ will be 1/4 of the source voltage when $R_1 = 9 \, \Omega$; then find the current flowing when $v_s = 12 \, V$

• Note, $v_s$, $R_1$ and $R_2$ are all in series with current $i$
Solution

• The voltage across resistor $R_2$ will be:

$$v_2 = \frac{R_2}{R_1 + R_2}v_s$$

• Since we desire $v_2/v_s = 1/4$, we have:

$$\frac{R_2}{R_1 + R_2} = \frac{1}{4} \quad \Rightarrow \quad R_1 = 3R_2$$

• Since $R_1 = 9 \ \Omega$, we require that $R_2 = 3 \ \Omega$. Using KVL around the loop, we have:

$$-v_s + v_1 + v_2 = 0 \quad \Rightarrow \quad v_s = iR_1 + iR_2$$

• Therefore:

$$i = \frac{v_s}{R_1 + R_2} = \frac{12}{12} = 1 \text{A}$$
Example 3.3-3: Voltage Divider Design

• Design a voltage divider by specifying values for $R_1$ and $R_2$ that satisfy the two specifications:
  – The input and output voltages are related by $v_o = 0.8 \, v_s$
  – The voltage source supplies no more than 1 mW of power when the input to the voltage divider is $v_s = 20$ V
Solution

• We'll examine each specification to see what it tells us about the resistor values:

1. The input and output voltages of the voltage divider are related by:

\[ v_o = \frac{R_2}{R_1 + R_2} v_s \]

– So specification 1 requires:

\[ \frac{R_2}{R_1 + R_2} = 0.8 \Rightarrow R_2 = 4R_1 \]
Solution

2. The power supplied by the voltage source is given by:

\[ P_s = i_s v_s = \left( \frac{v_s}{R_1 + R_2} \right)v_s = \frac{v_s^2}{R_1 + R_2} \]

– So specification 2 requires:

\[ 0.001 \geq \frac{20^2}{R_1 + R_2} \Rightarrow R_1 + R_2 \geq 400 \times 10^3 = 400 \text{k}\Omega \]

– Combining these results gives:

\[ 5R_1 \geq 400 \text{k}\Omega \]

– Since the solution is not unique, we choose:

\[ R_1 = 100 \text{k}\Omega \text{ and } R_2 = 400 \text{k}\Omega \]
Parallel Resistors and Current Division

- The defining characteristic of parallel elements is that they have the same voltage, i.e., two elements connected between the same pair of nodes.

- All elements are connected in parallel; thus the order of parallel resistors is not important.
Parallel Resistors and Current Division

- Consider the circuit with two resistors and a current source below, noting that both resistors are connected to terminals a and b and that the voltage $v$ appears across each parallel element.

- We may write KCL at node a (or at node b) to obtain:

$$i_s = i_1 + i_2$$
Parallel Resistors and Current Division

- From Ohm’s law: \[ i_1 = \frac{v}{R_1} \quad \text{and} \quad i_2 = \frac{v}{R_2} \]

\[ \Rightarrow \quad i_s = \frac{v}{R_1} + \frac{v}{R_2} \]

- Conductance \( G \) is defined as the inverse of resistance \( R \); therefore:
  \[ i_s = G_1 v + G_2 v = (G_1 + G_2) v \]

\[ \Rightarrow \quad G_p = G_1 + G_2 \]

- Thus, the equivalent circuit for the parallel circuit is:
Parallel Resistors and Current Division

• With an equivalent resistance given by:

\[
G_p = G_1 + G_2 = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_p}
\]

\[
R_p = \frac{R_1 R_2}{R_1 + R_2}
\]

• Note that the total conductance, \( G_p \), increases as additional parallel elements are added and that the total resistance, \( R_p \), decreases as each parallel resistor is added.

• Recall that the units for conductance are siemens (S).
Parallel Resistors and Current Division

- The circuit below is called a *current divider* since it divides the source current between conductances $G_1$ and $G_2$ in proportion to their conductance values.

$$i_s = i_1 + i_2 = \frac{G_1 i_s}{G_1 + G_2} + \frac{G_2 i_s}{G_1 + G_2} = \frac{R_2 i_s}{R_1 + R_2} + \frac{R_1 i_s}{R_1 + R_2}$$
Consider the more general case of current division with a set of $N$ parallel conductors as shown to the left:

\[ i_s = i_1 + i_2 + i_3 + \ldots + i_N \]
\[ = (G_1 + G_2 + G_3 + \ldots + G_N)v \]

\[ \Rightarrow \quad i_s = v \sum_{n=1}^{N} G_n = v G_p \]

Since $i_n = G_n v$, solving for $v$ we obtain the current divider eqn. with $N$ conductances:

\[ i_n = \frac{G_n i_s}{\sum_{n=1}^{N} G_n} = \frac{G_n i_s}{G_p} \]
Parallel Resistors and Current Division

• Since the equivalent conductance $G_p$ can be expressed as the sum of branch conductances

$$G_p = \sum_{n=1}^{N} G_n$$

with $G_n = 1/R_n$, the equivalent conductance $G_p$ can be expressed in terms of equivalent resistance $R_p$, as shown:

$$\frac{1}{R_p} = \sum_{n=1}^{N} \frac{1}{R_n}$$
Example 3.4-1: Parallel Resistors

• For the circuit below find (a) the current in each branch, (b) the equivalent circuit, and (c) the voltage $v$:

\[
R_1 = \frac{1}{2} \Omega, \quad R_2 = \frac{1}{4} \Omega, \quad R_3 = \frac{1}{8} \Omega
\]
Solution

• The current divider follows the equation:

\[ i_n = \frac{G_n i_s}{G_p} \]

where

\[ G_p = \sum_{n=1}^{N} G_n = G_1 + G_2 + G_3 = 2 + 4 + 8 = 14 \, \text{S} \]

then

\[ i_1 = \frac{G_1 i_s}{G_p} = \frac{2}{14} (28) = 4 \, \text{A} \]

\[ i_2 = \frac{G_2 i_s}{G_p} = \frac{4(28)}{14} = 8 \, \text{A} \]

\[ i_3 = \frac{G_3 i_s}{G_p} = 16 \, \text{A} \]
Solution

- Since $i_n = G_n v$, we have:
  \[ v = \frac{i_1}{G_1} = \frac{4}{2} = 2 \text{ V} \]

- And the equivalent circuit is given by:

  ![Equivalent Circuit Diagram]

  \[ R_p = \frac{1}{G_p} = \frac{1}{14} \Omega \]

- Note the equivalent circuit resistance is less than each of the individual branch resistances, as expected.
Example 3.4-2: Parallel Resistors

- For the circuit below find (a) the voltage measured by the voltmeter and (b) show that the power absorbed by the two resistors is equal to that supplied by the source:

\[
\begin{align*}
\text{Kirchhoff's Law:} & \quad 250 \text{ mA} + i_1 + i_2 = 0 \\
& \quad i_1 + i_2 = -250 \text{ mA}
\end{align*}
\]
Solution

• We obtain the equivalent circuit by combining the parallel resistors as shown below:

\[
\frac{1}{\frac{1}{40} + \frac{1}{10}} \Omega
\]

• The current in the equivalent resistor is 250 mA directed upward; since the current and the voltage references do not adhere to the passive convention, Ohm’s law gives:

\[
v_m = 8(-0.25) = -2 \text{ V}
\]
Solution

• The power absorbed by the resistors is:

\[ P_R = \frac{v_m^2}{40} + \frac{v_m^2}{10} = \frac{2^2}{40} + \frac{2^2}{10} = 0.1 + 0.4 = 0.5 \text{ W} \]

• The power supplied by the current source is:

\[ P_s = 2(0.25) = 0.5 \text{ W} \]

• The power absorbed by the two resistors is equal to that supplied by the source, as expected
Series Voltage and Parallel Current Sources

- Voltage sources connected in series are equivalent to a single voltage source equal to the sum of the series voltages sharing a common current:

![Circuit Diagrams]

- Series current sources are not allowed.
Series Voltage and Parallel Current Sources

- Current sources connected in parallel are equivalent to a single current source equal to the sum of the parallel currents sharing a common voltage:

\[ i_a + i_b = i \]

- Parallel voltage sources are not allowed.
Circuit Analysis

• In this section we consider the analysis of a circuit by replacing a set of resistors with an equivalent resistance, thus reducing the network to a form easily analyzed.

• Consider the circuit below includes a set of series resistors and another set of parallel resistors:
Circuit Analysis

To find the output voltage $v_o$, we reduce the circuit to the equivalent circuit shown below:

\[
Rs = R_1 + R_2 + R_3
\]

\[
R_p = \frac{1}{G_p} = \frac{1}{G_4 + G_5 + G_6} = \frac{1}{\frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}}
\]

\[
v_o = \frac{R_p}{R_s + R_p} \cdot v_s
\]
Example 3.6-3: Circuit Analysis Using Equivalent Resistances

- Determine the values of $i_3$, $v_4$, $i_5$, and $v_6$ in circuit shown to the left
- An equivalent circuit is shown below:

\[
\begin{bmatrix}
1 & 61 & 8 & 1 & 2 & 8 & 11 \\
24 & 12
\end{bmatrix} \Rightarrow \begin{cases}
18 \Omega \\
5 \Omega
\end{cases}
\]

where

\[
R_1 = \left[(6+18) \parallel 12\right] \Omega = \left[\frac{1}{\frac{1}{24} + \frac{1}{12}}\right] \Omega = 8 \Omega
\]
Example 3.6-3: Circuit Analysis Using Equivalent Resistances

- An equivalent circuit is shown below:

where

\[ R_2 = \left[ \left( \frac{1}{20} \right| \left( \frac{1}{5} \right) \right] + 12 \Omega = \left[ \frac{1}{\frac{1}{20} + \frac{1}{5}} \right] + 12 \Omega \]

\[ = \left[ 4 + 12 \right] \Omega = 16 \Omega \]
Example 3.6-3: Circuit Analysis Using Equivalent Resistances

- An equivalent circuit is shown below:

where

\[ R_3 = \left[ \frac{1}{\left( \frac{1}{2} + \frac{1}{6} \right)} \right] \Omega = \left[ \frac{1}{\frac{1}{8} + \frac{1}{8}} \right] \Omega = 4\Omega \]
Example 3.6-3: Circuit Analysis Using Equivalent Resistances

- From the equivalent circuit:

- By Ohm’s law:

\[
i = \left[ \frac{18V}{(8 + R_1 + R_2 + R_3) \Omega} \right] = \left[ \frac{18V}{(8 + 8 + 16 + 4) \Omega} \right] = \left[ \frac{18}{36} \right] A = 0.5 A
\]

\[
\Rightarrow v_{bd} = R_2i = 16 \Omega \cdot 0.5 A = 8 V
\]
Example 3.6-3: Circuit Analysis Using Equivalent Resistances

- From the equivalent circuit:

- By Voltage Divider:

\[
v_1 = \frac{18V \cdot R_1 \Omega}{(8 + R_1 + R_2 + R_3) \Omega} = \frac{18V \cdot 8\Omega}{(8+8+16+4)\Omega} = 4V
\]

\[
v_{bd} = \frac{18V \cdot R_2 \Omega}{(8 + R_1 + R_2 + R_3) \Omega} = \frac{18V \cdot 16\Omega}{(8+8+16+4)\Omega} = 8V
\]

\[
v_2 = \frac{18V \cdot R_3 \Omega}{(8 + R_1 + R_2 + R_3) \Omega} = \frac{18V \cdot 4\Omega}{(8+8+16+4)\Omega} = 2V
\]
Example 3.6-3: Circuit Analysis Using Equivalent Resistances

- We can now determine the values of $i_3$ and $i_5$ using Current Division, as shown:

$$i_3 = \frac{8\Omega \cdot iA}{8\Omega + (2 + 6)\Omega}$$
$$= \frac{8\Omega \cdot 0.5A}{8\Omega + (2 + 6)\Omega} = 0.25A$$

$$i_5 = -\left[\frac{5\Omega \cdot iA}{5\Omega + 20\Omega}\right]$$
$$= -\left[\frac{5\Omega \cdot 0.5A}{25\Omega}\right] = -0.1A$$
Example 3.6-3: Circuit Analysis Using Equivalent Resistances

- We can now determine the values of $v_4$ and $v_6$ using Voltage Division, as shown:

$$v_4 = - \left[ \frac{18\Omega \cdot v_1 V}{18\Omega + 6\Omega} \right]$$

$$= - \left[ \frac{18\Omega \cdot 4V}{24\Omega} \right] = -3V$$

$$v_6 = \left[ \frac{(20 \parallel 5)\Omega \cdot v_{bd} V}{12\Omega + (20 \parallel 5)\Omega} \right]$$

$$= \left[ \frac{4\Omega \cdot 8V}{16\Omega} \right] = 2V$$
Analyzing Resistive Circuits Using MATLAB

- MATLAB is software for making mathematical calculations, both numerical and symbolic
  - In this section MATLAB is used to solve the equations encountered when analyzing a resistive circuit
  - Consider the resistive circuit shown below where the goal is to determine the value of the input voltage, $V_s$, required to cause the current $I$ to be 1 A:
Analyzing Resistive Circuits Using MATLAB

• An equivalent circuit is given by:

\[ R_s = R_1 + R_2 + R_3 \]

• Resistors \( R_1, R_2, \) and \( R_3 \) are connected in series and can be replaced by an equivalent resistor, \( R_s \), given by:

• By Voltage Divider:

\[ V_o = \frac{R_p}{R_s + R_p} \cdot V_s \]

• By Ohm’s Law:

\[ I = \frac{V_o}{R_6} \]
Analyzing Resistive Circuits Using MATLAB

- A MATLAB program was written to vary $V_s$ over a range of voltages, calculate the value of $I$ corresponding to each value of $V_s$ using the circuit relations, then plot the current $I$ versus the voltage $V_s$
  - The resulting plot shows that $I$ will be 1 A when $V_s = 14$ V
% Analyzing Resistive Circuits Using MATLAB - ch3ex.m
% Vary the input voltage from 8 to 16 volts in 0.1 volt steps.
Vs = 8:0.1:16;

% Enter Values of the resistances.
R1 = 1; R2 = 2; R3 = 3; % series resistors, ohms
R4 = 6; R5 = 3; R6 = 2; % parallel resistors, ohms

% Find the current, I, corresponding to each value of Vs.
Rs = R1 + R2 + R3; % Equation 3.7-1
Rp = 1 / (1/R4 + 1/R5 + 1/R6); % Equation 3.7-2
for k = 1:length(Vs)
    Vo(k) = Vs(k) * Rp / (Rp + Rs); % Equation 3.7-3
    I(k) = Vo(k) / R6; % Equation 3.7-4
end

% Plot I versus Vs
plot (Vs, I)
grid
xlabel ('Vs, V'), ylabel ('I, A')
title ('Current in R6')
How Can We Check...?

• Engineers should verify that a solution to a problem is correct, e.g., solutions to design problem must be checked to confirm that all of the specifications have been satisfied.

• In addition, computer output must be reviewed to guard against data-entry errors while claims made by vendors must be examined critically.

• Engineering students can also check the correctness of their work, e.g., if some time remains at the end of an exam it is useful to quickly identify solutions that need more work.

• Generally speaking, all nodes must satisfy KCL, all branches must satisfy KVL, all resistive elements must obey Ohm’s Law, and power supplied must be equal to power absorbed by all elements in a circuit.
Example 3.8-1: Checking Values?

- The circuit shown was analyzed by writing and solving a set of simultaneous equations using a combination of KVL, KCL and Ohm’s Law:

\[
12 = v_2 + 4i_3, \quad i_4 = \frac{v_2}{5} - i_3, \quad v_5 = 4i_3,
\]

and

\[
\frac{v_5}{2} = i_4 + 5i_4 = 6i_4
\]

- These 4 equations in 4 “unkowns” can be solved numerically using matrix methods in MATLAB as shown on the following slide...
Example 3.8-1: Checking Values?
The 4 scalar eqns. can be written as:

\[ 1 \cdot v_2 + 4 \cdot i_3 + 0 \cdot i_4 + 0 \cdot v_5 = 12 \]
\[ 1 \cdot v_2 - 5 \cdot i_3 - 5 \cdot i_4 + 0 \cdot v_5 = 0 \]
\[ 0 \cdot v_2 + 4 \cdot i_3 + 0 \cdot i_4 - 1 \cdot v_5 = 0 \]
\[ 0 \cdot v_2 + 0 \cdot i_3 + 12 \cdot i_4 - 1 \cdot v_5 = 0 \]

Writing the eqns. in matrix form yields:

\[
\begin{bmatrix}
1 & 4 & 0 & 0 \\
1 & -5 & -5 & 0 \\
0 & 4 & 0 & -1 \\
0 & 0 & 12 & -1 \\
\end{bmatrix}
\begin{bmatrix}
v_2 \\
i_3 \\
i_4 \\
v_5 \\
\end{bmatrix}
= 
\begin{bmatrix}
12 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]
Example 3.8-1: Checking Values?

The matrix eqn. can be solved as:

\[ A \cdot x = b \implies x = A^{-1} \cdot b \]

This can be implemented using MATLAB as shown:

```matlab
>> A=[1,4,0,0;1,-5,-5,0;0,4,0,-1;0,0,12,-1];
>> b=[12;0;0;0];
>> x=A\b
```

\[ x = \begin{bmatrix} 7.5000 \\ 1.1250 \\ 0.3750 \\ 4.5000 \end{bmatrix} \]

Now verify the solution using KVL, KCL, Ohm's Law and conservation of energy, i.e., power supplied = power absorbed for all circuit elements.
Design Example: *Adjustable Voltage Source*

- A circuit is required to provide an adjustable voltage. The specifications for this circuit are:
  1. It should be possible to adjust the voltage to any value between −5 V and +5 V, but not be possible to accidentally obtain a voltage outside this range
  2. The load current will be negligible
  3. The circuit should use as little power as possible

- The available components are:
  1. Potentiometers: resistance values of 10 kΩ, 20 kΩ, and 50 kΩ are in stock
  2. A large assortment of standard 2 percent resistors having values between 10 Ω and 1 MΩ (see Appendix E)
  3. Two power supplies (voltage sources): one 12-V supply and one −12-V supply, both rated at 100 mA (maximum)
Describe the Situation and the Assumptions

• The figure below shows the situation: voltage $v$ is the adjustable voltage. Note, the circuit that uses the output of the circuit being designed is frequently called the “load.” In this case, the load current is negligible, so $i = 0$.

State the Goal

• A circuit providing the adjustable voltage

$$-5V \leq v \leq +5V$$

must be designed using the available components.
Generate a Plan

- We can now make the following observations:
  1. The adjustability of a potentiometer can be used to obtain an adjustable voltage $v$.
  2. Both power supplies must be used so that the adjustable voltage can have both positive and negative values.
  3. The terminals of the potentiometer cannot be connected directly to the power supplies because the voltage $v$ is not allowed to be as large as 12 V or $-12$ V.

- The observations suggest the following circuit:
Generate a Plan

• To complete the design, values need to be specified for $R_1$, $R_2$, and $R_p$. Then several results need to be checked and adjustments made, if necessary. Relevant questions:

1. Can the voltage $v$ be adjusted to any value in the range $-5 \text{ V}$ to $+5\text{V}$?

2. Are the voltage source currents less than 100 mA? This condition must be satisfied if the power supplies are to be modeled as ideal voltage sources.

3. Is it possible to reduce the power absorbed by $R_1$, $R_2$, and $R_p$?
Generate a Plan

• It seems likely that $R_1$ and $R_2$ will have the same value, so let $R_1 = R_2 = R$, which results in the following equivalent circuit:

• Applying KVL to the outside loop yields:

\[-12 + Ri_a + aR_p i_a + (1 - a)R_p i_a + Ri_a - 12 = 0\]

\[\implies i_a = \frac{24}{2R + R_p}\]
Generate a Plan

• Applying KVL to the left loop and substituting for $i_a$ yields:

\[ v = 12 - (R + aR_p)i_a = 12 - \frac{24(R + aR_p)}{2R + R_p} \]

• When the pot is adjusted so that $a = 0$, $v$ must be 5 V, so:

\[ 5 = 12 - \frac{24R}{2R + R_p} \]

\[ \Rightarrow R = 0.7R_p \]

• The power absorbed by the three resistances is:

\[ p = i_a^2(2R + R_p) = \frac{24^2}{2R + R_p} = \frac{24^2}{2(0.7R_p) + R_p} = \frac{240}{R_p} \]

• Clearly, the power absorbed is minimized when $R_p$ is maximized; therefore select $R_p = 50\, \text{k}\Omega$

\[ \Rightarrow R = 0.7 \times R_p = 35\, \text{k}\Omega \]
Verify the Proposed Solution

Since
$$-5 \, V = 12 - \left( \frac{35k + 50k}{70k + 50k} \right) 24 \leq V \leq 12 - \left( \frac{35k}{70k + 50k} \right) 24 = 5 \, V$$

the specification that
$$-5 \, V \leq V \leq 5 \, V$$
has been satisfied. The power absorbed by the three resistances is now
$$p = \frac{24^2}{50k + 70k} = 5 \, mW$$

Finally, the power supply current is
$$i_a = \frac{24}{50k + 70k} = 0.2 \, mA$$

which is well below the 100 mA that the voltage sources are able to supply.
Questions?