EE/ME/AE324: Dynamical Systems

Chapter 5: Modeling Rotational Mechanical Systems
Common Variables Used

- Assume 1 rotational DoF per mass, i.e., all motion scalar
- Angular Displacement: $\theta(t)$ [rad]
- Angular Velocity: $\omega(t) = \frac{d\theta(t)}{dt}$ [rad/s]
- Angular Acceleration: $\alpha(t) = \frac{d\omega(t)}{dt} = \frac{d^2\theta(t)}{dt^2}$ [rad/s$^2$]
- Torque: $\tau(t) = \frac{d(J(t)\omega(t))}{dt} = J \frac{d\omega(t)}{dt} = J\alpha(t)$ [N \cdot m]
- Energy (work): $w(t) = w(t_0) + \int_{t_0}^{t} p(\lambda) d\lambda$ [J or N \cdot m]
- Power: $p(t) = \frac{dw(t)}{dt} = \tau(t)\omega(t)$ [W or J/s]
Variable Conventions

- Position, velocity, acceleration, torque and supplied power:

\[ p = \tau \omega \]
Element Laws: Moment of Inertia

- Moment of Inertia (MoI): $J \text{ [kg} \cdot \text{m}^2 \text{]}$
  - Assumes constant, non-relativistic motion with 1 rotational DoF
    \[ \tau = J\alpha = J\dot{\omega} \]

- Obtained by integrating $r^2 dm$ over entire body
  - The MoI for a body of mass $M$ concentrated at a point is $ML^2$, where $L$ is the distance from the point to the axis of rotation
  - Simple cases (see MoI reference on class web site):
    \[
    J = \frac{1}{12}ML^2 \quad \text{and} \quad J = \frac{1}{2}MR^2
    \]
Parallel-Axis Theorem

- Applied to calculate MoI for rotations not about an object’s Center of Mass (CoM):

\[ J = J_0 + Md^2 \]

\( J_0 \) denotes the MoI about the parallel axis passing through the CoM, \( M \) is the total mass, and \( d \) is the distance between the parallel axes.

\[ J = \frac{1}{12} ML^2 + M \left( \frac{L}{2} \right)^2 = \frac{1}{3} ML^2 \]

\[ J = \frac{1}{3} M_1 d_1^2 + \frac{1}{3} M_2 d_2^2 = \frac{M \left( d_1^3 + d_2^3 \right)}{3 \left( d_1 + d_2 \right)} \]
Element Laws: Moment of Inertia

- Kinetic Energy: \( w_k = \frac{1}{2} J \omega^2 \)

- Potential Energy: \( w_p = Mgh \)
  
  - \( g = 9.81 \text{ [m/s}^2]\) @ surface of the earth
  
  - \( h \) is height of center of mass above(below) a specified reference point, e.g. surface of earth
Element Laws: Viscous Friction

- Viscous friction: \( \tau = B \Delta \omega = B(\omega_2 - \omega_1) \)

- In this case it is difficult to visualize a “stretched” versus a “compressed” component; therefore, we will note that for proper power conventions \( \tau \) and \( \omega_2 \) are in same direction when \( \omega_2 > \omega_1 > 0 \)

- Power supplied to damper \( p = \tau \Delta \omega > 0 \) lost as heat!
Element Laws: Viscous Friction

• How are rotational dampers built?
Element Laws: Stiffness (Spring)

- Stiffness: \( \tau = K \Delta \theta = K (\theta_2 - \theta_1) \)
  - \( \tau \) and \( \theta_2 \) are in the same direction when \( \theta_2 > \theta_1 > 0 \)

\[
\begin{align*}
\theta_1 & \quad \tau \\
& \quad \theta_2
\end{align*}
\]

- For thin shafts, stiffness constant \( K \) is:
  - directly proportional to material shear modulus and square of the cross-sectional area
  - inversely proportional to the length of the shaft

- Potential energy stored in the spring:

\[
w_p = \frac{1}{2} K (\Delta \theta)^2 > 0
\]
Element Laws: Levers

- Ideal lever is a rigid bar, pivoted at a point having no mass, friction, momentum or stored energy
- For small displacements $\theta$:

$$x_1 \approx d_1 \theta \text{ and } x_2 \approx d_2 \theta \Rightarrow x_2 = \left( \frac{d_2}{d_1} \right) x_1$$

$$\tau_1 = f_1 d_1 = f_2 d_2 = \tau_2 \Rightarrow f_2 = \left( \frac{d_1}{d_2} \right) f_1$$
Element Laws: Gears

- Ideal gear is a rigid bar, pivoted at a point having no MoI, friction or stored energy and perfect meshing of teeth, i.e., no backlash or slippage.

- I/O displacements and velocities are proportional to the gear sizes:

\[
\frac{\theta_1}{\theta_2} = \frac{r_2}{r_1} = \frac{n_2}{n_1} = N \text{ (gear ratio)}
\]
Element Laws: Gears

- I/O torques are proportional to the gear sizes:

\[
\frac{\tau_2}{\tau_1} = \frac{-r_2}{r_1} = -N
\]

Note: \( \tau_1 - r_1f_c = 0 \)

- Negative sign results since power assumed supplied to each gear by torque and displacement references
Interconnection Laws

- D’Alembert’s: \( \sum_i (\tau_{ext})_i - J\alpha = \sum_i \tau_i = 0 \)

- Law of Reaction Torques

- Law of Angular Displacements:
  \( \sum_i (\Delta \theta)_i = 0 \), around closed path

\[ \Delta \theta = \theta_2 - \theta_1 \]
Free-body Diagrams

• Free-body diagrams are used as an intermediate step to obtaining system equations of motion (EoM)
• Assume all elements at equilibrium (EQ) when position and velocity references equal to zero
  
  Equilibrium $\Rightarrow$ Net torque on body $= 0$,
  
  with all inputs constant (zero)

• Suggested order applying torques to a free-body diagram
  
  – Applied torques, i.e., specified inputs
  – Inertial torques, i.e., opposite position reference
  – Spring torques
  – Viscous friction (damping) torques
  – All others, e.g., gears, pulleys, levers, etc.
Simple Free-body Example 5.2

- Assume elements at equilibrium (EQ) when $\theta_2 = \theta_1 = 0$, and $\theta_2 > \theta_1 > 0$. See $\tau_{K2}$ and $\tau_{K1}$ in red below!
Ex. 5.2 Equations of Motion

- To obtain EoM from free-body diagram is straightforward
  \[ \sum \text{C.W. pointing torques} = \sum \text{C.C.W. pointing torques} \]
- Given the prior free-body diagram

We have the following EoM:

\[
\begin{align*}
J_2 \ddot{\theta}_2 + B_2 \dot{\theta}_2 + K_2 (\theta_2 - \theta_1) &= \tau_a(t) + K_2 \theta \\
J_1 \dddot{\theta}_1 + B_1 \dot{\theta}_1 + K_1 \theta_1 &= K_2 (\theta_2 - \theta_1)
\end{align*}
\]
Twist on Example 5.2

- State assumptions and add missing torques to FBD
Ex 5.4

Assume pt. A has no MoI and \( \theta > \theta_A > 0 \)

- Equations of Motion:

\[
K_2(\theta - \theta_A) - K_1 \theta_A = 0
\]

\[
J \ddot{\theta} + B \dot{\theta} + K_2(\theta - \theta_A) = \tau_a(t)
\]
Ex 5.4

• EoM:

\[ F_{\text{eoM}} = K_2 (\theta - \theta_A) - K_1 \theta_A = 0 \]

\[ J \ddot{\theta} + B \dot{\theta} + K_2 (\theta - \theta_A) = \tau_a(t) \]

• From the EoM at pt. A, we have:

\[ \theta_A = \left( \frac{K_2}{K_1 + K_2} \right) \theta \]

Substituting this for \( \theta_A \) in the EoM at J yields:

\[ J \ddot{\theta} + B \dot{\theta} + K_{eg} \theta = \tau_a(t), \text{ where } K_{eg} = \left( \frac{K_1 K_2}{K_1 + K_2} \right) \]

⇒ Series elements act the same way as before!
What about parallel elements?
Ex. 5.5: A Relative Displacement

- What could this system represent?

Find the state eqn. assuming system in EQ when states:

$$\dot{\theta}_1 = \dot{\theta}_2 = \theta_R = \theta_A - \theta_2 = 0$$

Also assume:

$$\theta_1 > \theta_A > \theta_2 = 0$$

$$\Rightarrow \theta_R > 0$$
Ex. 5.5: A Relative Displacement

\[ \tau_a(t) \]

\[ B(\omega_1 - \omega_A) \]

\[ K(\theta_A - \theta_2) \]

\[ J_1 \dot{\omega}_1 \]

\[ J_2 \dot{\omega}_2 \]

\[ \theta_2, \omega_2 \]

\[ \tau_L(t) \]
Ex. 5.5: A Relative Displacement

- EoM:

\[ J_1 \ddot{\theta}_1 + B \dot{\theta}_1 = B \dot{\theta}_A + \tau_a(t) \]
\[ B \dot{\theta}_1 = B \dot{\theta}_A + K(\theta_A - \theta_2) \]
\[ J_2 \ddot{\theta}_2 + \tau_L(t) = K(\theta_A - \theta_2) \]

- Since \( \theta_R = \theta_A - \theta_2 \Rightarrow \dot{\theta}_R = \dot{\theta}_A - \dot{\theta}_2 \), the state-eqns. are:

\[ \dot{\theta}_R = -\frac{K}{B} \theta_R + \dot{\theta}_1 - \dot{\theta}_2 \]
\[ \ddot{\theta}_1 = -\frac{K}{J_1} \theta_R + \frac{1}{J_1} \tau_a(t) \]
\[ \ddot{\theta}_2 = \frac{K}{J_2} \theta_R - \frac{1}{J_2} \tau_L(t) \]

Note: How is the load \( \tau_L(t) \) typically defined, e.g., for a fan?
Ex. 5.6: An Ideal Lever

- Find the state space representation of the system with:
  system input \( x_4(t) \) and output \( f_r \),
  the reaction force acting on the pivot
- Assume system at EQ when
  all displacements zero and
  angular displacement \( \theta \) remains small
- Assume as drawn: \( x_1, x_2 > 0, \ x_4(t) > x_3 > 0 \)
  \[ \Rightarrow K_1 \text{ compressed by } k(x_1 + x_2) \]
  \[ K_2 \text{ stretched by } (x_3 - x_3) \]
- For lever: \( x_3 = \left( \begin{array}{c} \frac{d_1}{d_2} \\ 1 \end{array} \right) x_2 \)
  \[ f_B = 8x_1 \]
  \[ f_{K_1} = k_1(x_1 + x_2) \]
Ex. 5.6: Free Body and EoM

Remember \( x_3 = \left( \frac{d_1}{d_2} \right) x_2 \)

\[
\begin{align*}
\dot{x}_1 + B \dot{x}_1 + K_1 (x_1 + x_2) &= 0 \\
K_2 (x_4(t) - x_3) d_1 &= K_1 (x_1 + x_2) d_2
\end{align*}
\]

No vectors.

\( K_2 (x_4(t) - x_3) d_1 = K_1 (x_1 + x_2) d_2 \) solve for \( x_1 \) in terms of \( x_2, x_4 \).
Ex. 5.6: State Representation

- If the desired system states are $x_1$ and $\dot{x}_1$, the level relations are used to eliminate $x_2$ and $x_3$ from the EoM to obtain:

\[
\dot{q} = \begin{bmatrix}
\dot{x}_1 \\
-\frac{1}{M} \left[ B\dot{x}_1 + \alpha K_1 x_1 + \alpha \left( \frac{d_2}{d_1} \right) K_1 x_4(t) \right]
\end{bmatrix}
\]

where $\alpha^{-1} = 1 + \frac{K_1}{K_2} \left( \frac{d_2}{d_1} \right)^2$

- The output $f_r$ can be found from the free-body as:

\[
y = f_r = K_2 \left( x_4(t) - x_3 \right) + K_1 \left( x_1 + x_2 \right) = c_{11} x_1 + d_{11} x_4(t)
\]
Ex. 5.6: Some Observations

• When lever pivoted at center, system reduces to:

\[ M \ddot{x}_1 + B \dot{x}_1 + K_{eq} (x_1 + x_4) = 0 \]

• If lever has a MoI and friction, then free-body is modified as:

\[ J \ddot{\theta} + B \dot{\theta} + K_1 (x_1 + x_2) \frac{d\theta}{dt} - K_1 (x_1 + x_2) d_2 \]

\[ = K_2 (x_4 - x_3) d_1 \]
Ex. 5.10: Ideal Gears

• Find the state space representation of the system:

\[ \tau_{B_1} = B_1 \dot{\theta}_1 \]

Treat all rotations from a common perspective!

\[ \theta_2 \approx \text{Counter-clockwise (CCW) or down} \]

\[ \theta_1 \approx \text{Clockwise (CW) or up} \]

• Assume ideal gears such that:

\[ \theta_1 = N \theta_2 \Rightarrow \omega_1 = N \omega_2 \text{ where } N = \frac{r_2}{r_1} < 1 \text{ as drawn,} \]

system in EQ when \( \theta_1 = \theta_2 = 0 \), and \( \theta_1, \theta_2 > 0 \) as drawn
Ex. 5.10: Free-body Diagram

- Direction of contact forces arbitrary so long as equal but opposite
Ex. 5.10: EoM

- From the Free-body Diagram:
  \[ J_1 \ddot{\theta}_1 + B_1 \dot{\theta}_1 + K_1 \theta_1 + r_1 f_c = \tau_{a1}(t) \]
  \[ J_2 \ddot{\theta}_2 + B_2 \dot{\theta}_2 + K_2 \theta_2 - r_2 f_c = \tau_{a2}(t) \]

- If the desired state variables are \( \dot{\theta}_2 \) and \( \theta_2 \), the gear ratio \( N \) can be used to combine the two EoM, eliminating \( \dot{\theta}_1 \) and \( \theta_1 \), as shown:
  \[ J_{2eq} \ddot{\theta}_2 + B_{2eq} \dot{\theta}_2 + K_{2eq} \theta_2 = N \tau_{a1}(t) + \tau_{a2}(t) \]
  where \( J_{2eq} = J_2 + N^2 J_1 \), \( B_{2eq} = B_2 + N^2 B_1 \), etc.
Ex. 5.10: Some Observations

• If the desired state variables were $\dot{\theta}_1$ and $\theta_1$, the gear ratio $N$ can be used to combine the two EoM, eliminating $\dot{\theta}_2$ and $\theta_2$, as shown:

$$J_{1eq} \ddot{\theta}_1 + B_{1eq} \dot{\theta}_1 + K_{1eq} \theta_1 = \tau_{a1}(t) + \frac{1}{N} \tau_{a2}(t)$$

where $J_{1eq} = J_1 + \frac{J_2}{N^2}$, $B_{1eq} = B_1 + \frac{B_1}{N^2}$, etc.

• Since $N < 1$ as drawn, the gears:
  reduce torques associated with mass 1
  increase torques associated with mass 2
Ex. 5.11: Rack and Pinion

- Assume system in EQ when $\theta = x = R\theta_A = 0$, that $\theta > \theta_A = \frac{x}{R} > 0$ (as drawn), and pinion gear has no MoI.

$$
\tau_K = K(\theta - \theta_A) \\
\tau_B = B_1(\dot{\theta} - \dot{\theta}) = B_1\dot{\theta}
$$
Ex. 5.11: Free-body
Ex. 5.11: EoM

- From the free-body diagrams we obtain:

\[ J\ddot{\theta} + B_1\dot{\theta} + K(\theta - \theta_A) = \tau_a(t) \]
\[ Rf_c = K(\theta - \theta_A) \]
\[ M\ddot{x} + B_2\dot{x} = f_c \]

- If the desired state variables are \( \dot{\theta}, \dot{x} \) and \( \theta_R = \theta - \theta_A = \theta - \frac{x}{R} \), eliminating \( \theta_A \) and \( f_c \) from the EoM yields the state eqns.:

\[ \dot{\theta}_R = \dot{\theta} - \frac{\dot{x}}{R} \]
\[ \ddot{\theta} = -\frac{K}{J} \theta_R - \frac{B_1}{J} \dot{\theta} + \frac{1}{J} \tau_a(t) \]
\[ \ddot{x} = -\frac{B_2}{M} \dot{x} - \frac{K}{MR} \theta_R \]
Ex. 5.12: Simple Elevator

- Find the state space representation of the system:
  
  Assume system in EQ when \( x = y = R\theta = 0 \),
  \( x > y = R\theta > 0 \) as drawn
  \( \Rightarrow K_1 \) "stretched" by \( \theta \)
  \( K_2 \) "stretched" by \( x - y = x - R\theta \)

Remember: The force/torques shown on the schematic, e.g., \( f_{K_2} \), are associated with the element, e.g., the spring, not the reaction forces/torques on the free-body diagrams.
Ex. 5.12: Free-bodies and EoM

\[ \begin{align*}
 &B \omega \\
 &K_1 \theta \\
 &J \dot{\omega} \\
 &R f_{K_2} = K_2 R(x - R \theta)
\end{align*} \]

- EoM:

\[ \begin{align*}
 J \ddot{\theta} + B \dot{\theta} + K_1 \theta &= K_2 R(x - R \theta) \\
 M \ddot{x} + K_2 (x - R \theta) &= Mg + f_a(t)
\end{align*} \]
Ex. 5.12: Some Observations

- Find static displacements due to gravity

\[ \ddot{\theta} = \dot{\theta} = \ddot{x} = f_a(t) = 0 \text{ in the EoM} \]

\[ \Rightarrow \theta_0 = \left[ \frac{RK_2}{K_1 + R^2K_2} \right] x_0 \]

\[ \Rightarrow x_0 = \frac{Mg \left( K_1 + R^2K_2 \right)}{K_2K_1} \]
Ex. 5.7: A Pendulum

• Find a state space representation of the system:

Identify torques acting upon the system?
Ex. 5.7: A Pendulum

EoM: \[ ML^2 \ddot{\theta} + B \dot{\theta} + MgL \sin \theta = \tau_a(t) \]

How does this system differ from those we've studied thus far?
It can be shown that the period of a simple pendulum is:

\[ T = 2\pi \sqrt{\frac{L}{g}} \]

\[ \Rightarrow L = g \left( \frac{T}{2\pi} \right)^2 \]

\[ = 10 \left( \frac{2.17}{2\pi} \right)^2 \approx 1 \text{ m} \]
Questions?
Exam 1 Results

EE324 Exam 1 Grades, 2/26/10

No. Students = 38
Mean = 89

No. Students = 7
Mean - Std = 77

No. Students = 9