Boundary Conditions in Fluid Mechanics

R. Shankar Subramanian
Department of Chemical and Biomolecular Engineering
Clarkson University

The governing equations for the velocity and pressure fields are partial differential equations that are applicable at every point in a fluid that is being modeled as a continuum. When they are integrated in any given situation, we can expect to see arbitrary functions or constants appear in the solution. To evaluate these, we need additional statements about the velocity field and possibly its gradient at the natural boundaries of the flow domain. Such statements are known as *boundary conditions*. Usually, the specification of the pressure at one point in the system suffices to establish the pressure fields so that we shall only discuss boundary conditions on the velocity field here. For a more detailed discussion of various aspects, the reader is encouraged to consult either Leal (1) or Batchelor (2).

Conditions at a rigid boundary

It is convenient for the purpose of discussion to identify two types of boundaries. One is that at the interface between a fluid and a rigid surface. At such a surface, we shall require that the tangential component of the velocity of the fluid be the same as the tangential component of the velocity of the surface, and similarly the normal component of the velocity of the fluid be the same as the normal component of the velocity of the surface. The former is known as the "no slip" boundary condition, and has been found to be successful in describing most practical situations. It was a subject of controversy in the eighteenth and nineteenth centuries, and was finally accepted because predictions based on assuming it were found to be consistent with observations of macroscopic quantities such as the flow rate through a circular capillary under a given pressure drop. If we designate the velocity of the rigid surface as $\boldsymbol{V}$ and that of the fluid as $\boldsymbol{v}$, and select a unit normal vector to the surface pointing into the fluid at a given location as $\boldsymbol{n}$, the no-slip boundary condition can be stated as

$$\boldsymbol{v} - (\boldsymbol{n} \bullet \boldsymbol{v})\,\boldsymbol{n} = \boldsymbol{V} - (\boldsymbol{n} \bullet \boldsymbol{V})\,\boldsymbol{n} \qquad \text{on a rigid surface} \qquad \text{(no slip)}$$

When there is no mass transfer across the boundary, a purely kinematical consequence is that the normal component of the fluid velocity at the boundary must equal the normal component of the velocity of the rigid surface.

$$\boldsymbol{v} \bullet \boldsymbol{n} = \boldsymbol{V} \bullet \boldsymbol{n} \qquad \text{on a rigid surface} \qquad \text{(kinematic condition)}$$

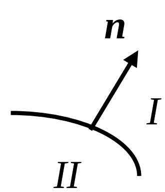
As a consequence of the two conditions, we arrive at the conclusion that the fluid velocity must be equal to the velocity of the rigid surface at every point on it.

$$\boldsymbol{v} = \boldsymbol{V} \qquad \text{on a rigid surface}$$

The no-slip condition has been found to be inapplicable in special circumstances such as at a moving contact line when a drop spreads over a solid surface, or in flow of a rarefied gas through a pore of diameter of the same order of magnitude as the mean free path of the gas molecules. For the types of problems that we shall encounter, it is an adequate boundary condition.

Conditions at a fluid-fluid interface

Sometimes, we encounter a boundary between two fluids. A common example occurs when a liquid film flows down an inclined plane. The surface of the liquid film in contact with the surrounding gas is a fluid-fluid interface. Other examples include the interface between a liquid drop and the surrounding continuous phase or that between two liquid layers. It is convenient to designate the two fluid phases in contact as phase I and phase II.



The unit normal vector $\boldsymbol{n}$ points into phase I here.

It so happens that the velocity fields in fluids I and II are continuous across the interface so long as there is no mass transfer across the interface. This vector condition also can be viewed as being in two parts, one on the continuity of the tangential component of the two velocities, analogous to the no-slip boundary condition at a rigid boundary.

$$\boldsymbol{v}_I - \left(\boldsymbol{n} \bullet \boldsymbol{v}_I\right)\boldsymbol{n} = \boldsymbol{v}_{II} - \left(\boldsymbol{n} \bullet \boldsymbol{v}_{II}\right)\boldsymbol{n} \qquad \text{at a fluid-fluid interface (continuity of tangential velocity)}$$

If the interface is in motion, we can describe it using the equation $F\left(t, x, y, z\right) = 0$ where t is time and F represents some function of time and position represented in Cartesian coordinates here just for convenience. Because $F = 0$ on the interface at all times, the derivative with respect to time following a material particle on the interface, also known as the material derivative, must be zero. That is,

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \boldsymbol{v} \bullet \nabla F = 0 \text{ on the interface.}$$

Assuming that F is defined such that ∇F is directed into phase I, the unit normal shown in the sketch is given by $\boldsymbol{n} = \nabla F / \left|\nabla F\right|$ so that $\boldsymbol{v} \bullet \nabla F = \boldsymbol{v} \bullet \boldsymbol{n}\left|\nabla F\right|$ where the velocity $\boldsymbol{v}$ is that of the interface. In the absence of mass transfer, the normal velocity is continuous across the interface and equal to the velocity of the interface normal to itself, as a kinematical consequence. Thus, the boundary condition on the normal velocity may be stated as follows.

$$\boxed{\boldsymbol{v_I} \bullet \boldsymbol{n} = \boldsymbol{v_{II}} \bullet \boldsymbol{n} = -\frac{1}{|\nabla F|} \frac{\partial F}{\partial t}}$$ at a fluid-fluid interface (kinematic condition)

If the interface is fixed in space, F is independent of time so that $\frac{\partial F}{\partial t} = 0$. In this case the above result simplifies to

$$\boxed{\boldsymbol{v_I} \bullet \boldsymbol{n} = \boldsymbol{v_{II}} \bullet \boldsymbol{n} = 0}$$ at a fluid-fluid interface

Notice that we have two unknown vector fields $\boldsymbol{v}_I$ and $\boldsymbol{v}_{II}$ now, and therefore need twice as many boundary conditions. Therefore, it is not sufficient to write just the above no-slip and kinematic conditions at a fluid-fluid interface. We also need to write a boundary condition connecting the state of stress in each fluid at the interface. This boundary condition is obtained from the principle that the forces on an element of interfacial area of arbitrary shape and size must be in equilibrium because the interface is assumed to have zero thickness and therefore zero mass. The following derivation is based on the development given by Leal (1).

Let us use the symbol A to designate the area of an interfacial element of some arbitrary shape and size, and C to designate the closed curve that forms its boundary. Let the local normal to C be labeled $\boldsymbol{n}_C$. This normal vector to C lies on the tangent plane to the surface at each point. Then, the condition that the total force on the area element must be zero can be written as follows.

$$\int_A \boldsymbol{n} \bullet \left[\boldsymbol{T}_I - \boldsymbol{T}_{II} \right] dA + \int_C \sigma \, \boldsymbol{n}_C \; ds = \boldsymbol{0}$$

In the above equation, the symbols $\boldsymbol{T_I}$ and $\boldsymbol{T_{II}}$ represent the stress tensor in each fluid, $\boldsymbol{n}$ is the unit normal pointing into fluid I, dA is a differential area element in A, σ is the interfacial tension, which can depend on position, and ds is a differential arc length on the closed curve C. Using a version of Stokes theorem, the line integral can be converted to an area integral, permitting us to rewrite the above balance as

$$\int_A \left(\boldsymbol{n} \bullet \left[\boldsymbol{T}_I - \boldsymbol{T}_{II} \right] + \nabla_s \sigma - \sigma \, \boldsymbol{n} \left\{ \nabla \bullet \boldsymbol{n} \right\} \right) dA = \boldsymbol{0}$$

In this result, ∇_s is the surface gradient operator which can be written as $\nabla - \boldsymbol{n}\left(\boldsymbol{n} \bullet \nabla\right)$. That is, we remove the part of the gradient vector that is normal to the surface. Because the integrand is a continuous function of position on the interface and the interfacial element chosen is of arbitrary size and shape, we can show that the integrand must be zero at every point on the interface. If any component of the vector integrand is non-zero at any point, then there must exist a neighborhood of that point on the interface in which that component of the integrand would be of the same sign, and the integration could be performed over that neighborhood to yield a non-zero result for that component of the force, thus violating the above equation. Therefore, we obtain the following result, which must be satisfied at every point on a fluid-fluid interface.

$$\boldsymbol{n} \bullet \left[\boldsymbol{T}_I - \boldsymbol{T}_{II} \right] = \sigma \, \boldsymbol{n} \left(\nabla \bullet \boldsymbol{n} \right) - \nabla_s \sigma$$

The divergence of the unit normal is related to the mean curvature H of the interface.

$$\nabla \bullet \boldsymbol{n} = 2H = \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where R_1 and R_2 are the principal radii of curvature of the interface at a given point. For more details about surface geometry, the reader may wish to consult a book such as that by Weatherburn (3). Using the above result, the following equation, called the jump condition on the stress, at a fluid-fluid interface, can be written.

$$\boldsymbol{n} \bullet \left[\boldsymbol{T}_I - \boldsymbol{T}_{II} \right] = 2H\sigma \boldsymbol{n} - \nabla_s \sigma \quad \text{at a fluid-fluid interface} \qquad \text{(jump condition on the stress)}$$

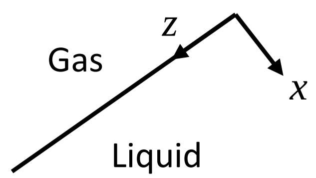
To summarize, in the stress boundary condition, the symbols $\boldsymbol{T_I}$ and $\boldsymbol{T_{II}}$ represent the stress tensor in each fluid, H is the mean curvature of the interface at the point where the condition is being applied, σ is the interfacial tension of the fluid-fluid interface, and ∇_s is the surface gradient operator. The left side in the stress boundary condition is the difference between the stress vectors in fluids I and II at the interface, or the "jump" in stress. This is the reason for the choice of terminology used in describing this condition. The resulting vector is decomposed into a part that is normal to the interface, namely the first term in the right side, and a part that is tangential to the interface, given in the second term in the right side. Sometimes, the condition is written as two separate scalar boundary conditions by writing the tangential and the normal parts separately. In that case, we call the two boundary conditions the "tangential stress balance" and the "normal stress balance."

Tangential Stress Balance

In the types of problems that we shall encounter, the stress boundary condition can be simplified. The interfacial tension at a fluid-fluid interface depends on the temperature and the composition of the interface. If we assume these to be uniform, then the gradient of interfacial tension will vanish everywhere on the interface. This means that the tangential stress is continuous across the interface because the jump in it is zero. Recall that the tangential stress is purely viscous in origin. If τ_t represents this stress component, we can write

$\tau_{t_I} = \tau_{t_{II}}$ at a fluid-fluid interface (tangential stress balance)

At a liquid-gas interface, we can further simplify the tangential stress balance. Consider the surface of a liquid film flowing down an inclined plane. Let us assume that the flow is steady and that the film surface is parallel to the inclined plane. In this situation, the normal velocity at the free surface of the liquid is zero in both the liquid and the gas. The sketch depicts the situation.



Because the normal velocity is zero at the free surface, the tangential stress balance simplifies to the following result where the subscripts l and g represent the liquid and gas, respectively.

$$\mu_l \frac{\partial v_{z,l}}{\partial x} = \mu_g \frac{\partial v_{z,g}}{\partial x} \quad \text{at the free surface}$$

The symbol μ in the above result stands for the dynamic viscosity. If we divide through by the dynamic viscosity of the liquid, we obtain

$$\frac{\partial v_{z,l}}{\partial x} = \frac{\mu_g}{\mu_l} \frac{\partial v_{z,g}}{\partial x} \quad \text{at the free surface}$$

Because the dynamic viscosity of a gas is small compared with that of a liquid, the right side of the above equation is small, and can be considered negligible. This allows us to write

$$\frac{\partial v_{z,l}}{\partial x} \approx 0 \quad \text{at the free surface}$$

Sometimes, this condition is represented as that of vanishing shear stress at a free liquid surface. Note that this approximation of the tangential stress condition can be used only when the motivating force for the motion of the liquid is not the motion of the gas. When a gas drags a liquid along, as is the case on a windy day when the wind causes motion in a puddle of liquid, the correct boundary condition equating the tangential stresses must be used.

Normal Stress Balance

The normal stress jump boundary condition actually determines the curvature of the interface at the point in question, and therefore the shape of the entire fluid-fluid interface. This shape is distorted by the flow. Thus, the boundary condition must be applied at an unknown boundary whose shape must be obtained as part of the solution. Fluid mechanical problems involving the application of the normal stress balance at a boundary are complicated, and must be solved numerically unless one assumes the shape distortion to be very small or of a particularly simple form.

For an illustration of a rather simple application of the balance of normal stress, consider a stationary system with no flow, so that $\boldsymbol{v}_I = \boldsymbol{v}_{II} = \mathbf{0}$. If we assume the surface tension to be uniform, the jump condition on the stress reduces to just a balance of normal stresses as follows.

$$\boldsymbol{n} \bullet \left[\boldsymbol{T}_I - \boldsymbol{T}_{II} \right] = 2H\sigma \boldsymbol{n}$$

The stress tensor in stationary fluids is simple. $\boldsymbol{T}_I = -p_I \boldsymbol{I}$ and $\boldsymbol{T}_{II} = -p_{II} \boldsymbol{I}$, where p is the pressure, so that the left side becomes $\left(p_{II} - p_I \right) \boldsymbol{n}$. Thus, the normal stress balance reduces to the scalar result

$$p_{II} - p_I = 2H\sigma = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

which is well-known. Consider a spherical liquid drop or gas bubble that is stationary inside another fluid. For a sphere of radius R, the curvature is uniform, so that $R_1 = R_2 = R$. Neglecting hydrostatic variation of pressure, we can then write

$$\boxed{p_{II} - p_I = \frac{2\sigma}{R}}$$

a result that relates the excess pressure within a spherical bubble or drop to the surface tension and the radius.

In the problems that we shall analyze, we shall always assume the shape of the interface to be the static shape and as being specified. Therefore, we shall not be able to necessarily satisfy the balance of normal stress.

References

1. L.G. Leal, “Laminar Flow and Convective Transport Processes,” Butterworth-Heinemann, 1992.

2. G. K. Batchelor, “An Introduction to Fluid Dynamics,” Cambridge University Press, 1967.

3. C.E. Weatherburn, “Differential Geometry of Three Dimensions,” Cambridge University Press, 1961.