The Equation of Conservation of Mass

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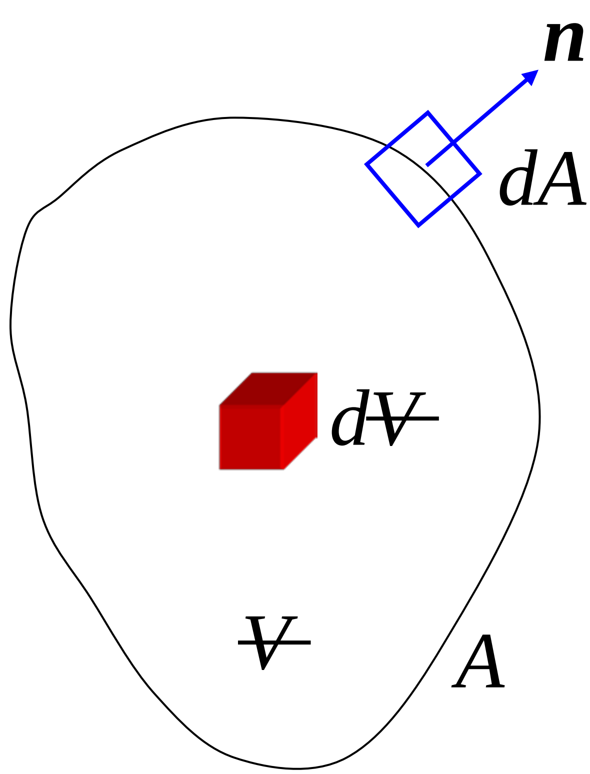
Based on observation, one can postulate the idea that mass is neither created nor destroyed. In other words, it is conserved. This is termed the Principle of Conservation of Mass. This principle is applied to a fixed volume in space of arbitrary shape that contains fluid. This volume is called a “Control Volume.” Fluid is permitted to enter or leave the control volume.

A control volume $\mathcal{V}$ is shown in the sketch.



We also have marked the bounding surface A of this control volume, called the control surface (CS) and shown an element of surface area dA and the unit outward normal (vector) to that area element, $\boldsymbol{n}$.

One can make similar statements about energy and momentum, being careful to accommodate ways in which energy or momentum can enter or leave a fixed volume in space occupied by a fluid. These conservation statements are put in mathematical form and termed “integral balances.” These balances include statements of conservation of mass, energy, and momentum, and will prove useful in a variety of problems. For example, conservation of mass allows us to estimate the rate of change of the level of liquid in a process vessel or the rate at which the amount of gas left in a tank decreases due to leakage. Conservation of mass and energy allow us to size pumps and turbines, and help in the evaluation of flow rates using flow measurement devices. Conservation of momentum is used in calculating the forces on the supports used for pipe bends and the forces flange bolts need to withstand. Also, by using these balances together, one can calculate the losses in a sudden expansion, design a jet ejector, and make calculations involving pipe manifolds.

Now, we shall proceed to express the idea that the total mass in the control volume that is shown on page 1 is conserved.

Rate of increase of mass of material within the control volume = Net rate at which material enters the control volume.

Let us write a mathematical representation of the above statement. If we designate the total mass of material in the control volume as M and the net rate of entry of mass into the control volume as $\dot{m}$, conservation of mass can be written as

$$\frac{dM}{dt} = \dot{m} \qquad (1)$$

Now, we need to work out suitable results for the left and right sides of the above equation. If we consider a differential volume $d\mathcal{V}$, the mass of fluid in that volume is obtained by multiplying the volume by the local density at that point ρ. By adding up all the differential volumes within the total volume $\mathcal{V}$, we can obtain the total mass of fluid M. Therefore, we can write

$$M = \int_{\mathcal{V}} \rho \, d\mathcal{V} \qquad (2)$$

The time rate of change of this mass is then $\frac{dM}{dt}$. Therefore,

$$\frac{dM}{dt} = \frac{d}{dt} \int_{\mathcal{V}} \rho \, d\mathcal{V} \qquad (3)$$

Now, we need to develop a result for the net rate of entry of fluid into the control volume through the control surface. For this, we consider the differential area element dA. If the velocity vector is $\boldsymbol{V}$, the component of this velocity that is directed **into** the control volume is given by $-\boldsymbol{V} \bullet \boldsymbol{n}$, because the unit normal vector $\boldsymbol{n}$ points **outward** from the control volume. This result, multiplied by the area of the element dA, gives the volumetric rate at which fluid enters the control volume through this area element, labeled dQ.

$$dQ = -\boldsymbol{V} \bullet \boldsymbol{n}\, dA \tag{4}$$

The corresponding rate at which mass enters the control volume through the area element dA, labeled $d\dot{m}$, can be written as

$$d\dot{m} = \rho\, dQ = -\rho\left(\boldsymbol{V} \bullet \boldsymbol{n} \right) dA \tag{5}$$

Adding up all the contributions from the differential area elements, which implies integration over the entire control surface, leads to a result for the net rate of entry of mass into the control volume.

$$\dot{m} = -\int_{CS} \rho\left(\boldsymbol{V} \bullet \boldsymbol{n} \right) dA \tag{6}$$

Now, we can rewrite the principle of conservation of mass, given in Equation (1) as

$$\frac{dM}{dt} = -\int_{CS} \rho \left(\boldsymbol{V} \bullet \boldsymbol{n}\right) dA \tag{7}$$

Rate of increase of mass M in the control volume	Rate of net inflow of mass into the control volume

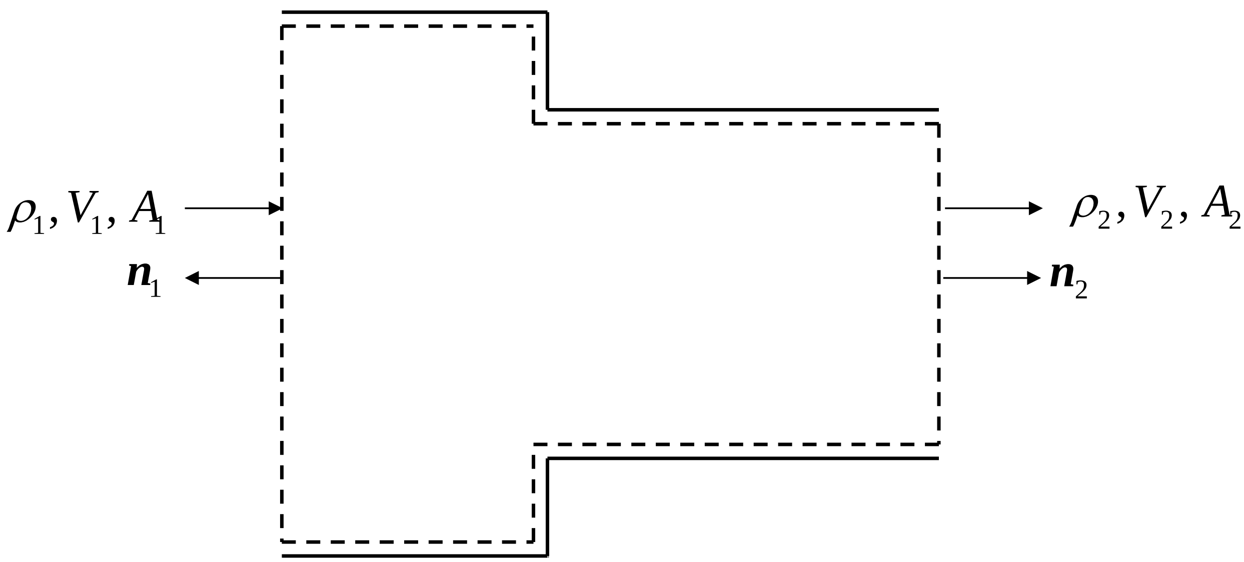
where we have identified the physical meaning of the terms in the left and right sides of the equation.

In a steady state situation, the time rate of change of the mass of material in a control volume is zero. In this case, we simply obtain

$$\int_{CS} \rho \left(\boldsymbol{V} \bullet \boldsymbol{n}\right)\, dA = 0 \tag{8}$$

Physically, this result implies that the influx of mass into the control volume must equal the efflux of mass at steady state.

Here is an example of how we may use this statement of "Conservation of Mass" at steady state. A fluid is in steady flow through a pipe of changing cross-section as shown in the sketch.



Location 1 is at the inlet and location 2 is at the outlet. Assume the velocity profiles are flat. The control volume is indicated by the dashed boundary. It is shown as being slightly separated from the physical boundary only for clarity. In reality, its surface coincides with the physical surface of the pipe. Note that there is no flow through most of the control surface, that is, $\boldsymbol{V} \bullet \boldsymbol{n} = 0$. There is flow only at the inlet (1) and exit (2). At the inlet surface the velocity points in a direction opposite to that of the normal vector. Therefore, $\boldsymbol{V} \bullet \boldsymbol{n}$ becomes $-V_1$. In a like manner, at the exit surface, the velocity points in the same direction as the normal vector which leads to $\boldsymbol{V} \bullet \boldsymbol{n}$ becoming V_2. Because V_1 and V_2 are assumed to be constant across the surfaces involved, the integrals are easy to evaluate and we get

$$- \rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0 \qquad (9)$$

or

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \qquad (10)$$

The product of the uniform velocity and the area is the volumetric flow rate Q. Therefore, we can rewrite this as

$$\rho_1 Q_1 = \rho_2 Q_2 \qquad (11)$$

The mass flow rate $\dot{m} = \rho\, Q$ so that $\dot{m} = \dot{m}_1 = \dot{m}_2$ is constant. If the density is constant, the above relationship reduces to $Q_1 = Q_2$. This assumption of constant density is known as the assumption of "incompressible flow." Even though the concept of incompressibility refers to changes in density associated with pressure changes, the term is used loosely to signify "constant density."

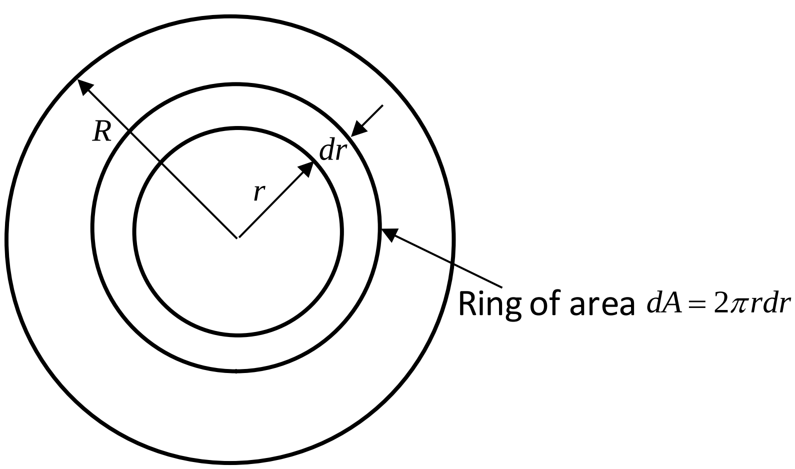
In isothermal single-component liquids, the assumption of constant density is nearly always an excellent one to make. In the case of gases flowing at velocities that are small compared with the speed of sound in the gas, it continues to be a good assumption. When density changes in a fluid caused by pressure, temperature, or composition variations are significant when compared with the average density, a precise calculation must accommodate such density variations.

Accommodating Velocity Variation Across the Cross-Section

In the simplification of the principle of conservation of mass, we assumed the velocity profile to be flat at the inlet and the exit. Realistically, the velocity of a fluid satisfies the no-slip condition at a solid boundary, and varies across the cross-section. This variation of velocity can be easily accommodated by using an average velocity V_{av} in Equations (9) and (10). The average velocity across the cross-section is defined as follows.

$$V_{av} = \frac{Q}{A} = \frac{\int_A \boldsymbol{V} \bullet \boldsymbol{n}\, dA}{\int_A dA} = \boxed{\frac{1}{A}\int_A \boldsymbol{V} \bullet \boldsymbol{n}\, dA} \qquad (12)$$

In Equation (12), the cross-sectional area A is oriented normal to the direction of flow, and $\boldsymbol{n}$ is a unit normal to the area in the direction of flow. The symbol dA stands for a differential area element. As an example, for a circular tube of radius R, in which the velocity distribution $V(r)$ is symmetric about the tube axis, the cross-section and differential area $dA = 2\pi r dr$ are illustrated in the sketch below.



Applying the definition given in Equation (12) yields

$$V_{av} = \frac{Q}{\pi R^2} = \frac{1}{\pi R^2}\int_0^R 2\pi r V(r)\,dr = \boxed{\frac{2}{R^2}\int_0^R r V(r)\,dr} \qquad (13)$$

Two example problems are considered next. The first example involves using the steady version of the equation of conservation of mass to make calculations involving a natural gas pipeline (source: Fluid Mechanics for Chemical Engineers by Noel de Nevers). The second example shows how to use the unsteady version of the equation of conservation of mass to calculate the rate of change of the height of liquid in a tank.

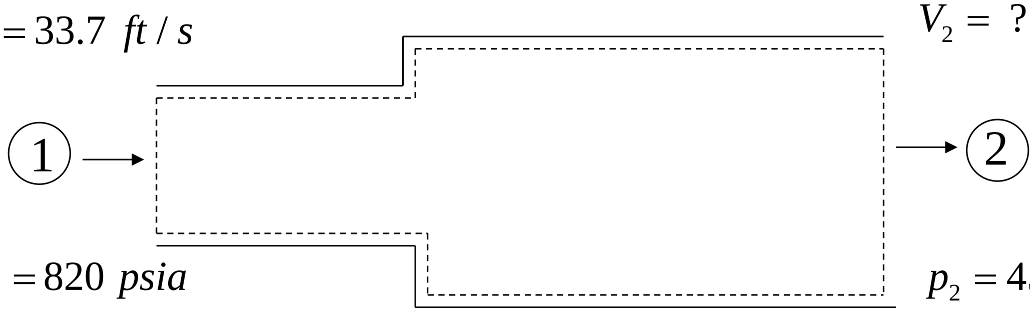
Example 1 -- Natural Gas Pipeline

$D_1 = 2.00\ ft$

$V_1 = 33.7\ ft/s$

$D_2 = 2.50\ ft$

$V_2 = \ ?$



$p_1 = 820\ psia$

$T_1 = 70^\circ\ F$

$p_2 = 450\ psia$

$T_2 = 58^\circ\ F$

Gas Constant $R_g = 10.73 \frac{\left(lb_f / in^2\right) \bullet ft^3}{\left(lb\ mole\right) \bullet {}^\circ R}$ Molecular Weight = $16 \frac{lb}{lb\ mole}$

Find $\dot{m}_2,\ V_2$

This is a long pipeline, which is shown schematically. The natural gas flows through a section of circular pipe of diameter 2.00 feet for some distance, and then through a larger section of circular pipe of diameter 2.50 feet. The pressure and temperature at locations 1 and 2 are specified. We need to find the velocity at location 2, and the mass flow rate at location 2.

At steady state, the mass flow rate remains the same throughout the pipeline because there is no accumulation of mass within the section of the pipeline shown with time. Therefore, the mass flow rate is the same at locations 1 and 2. The velocities are actually averages across each cross-section. The control volume is the interior of the pipe from the inlet at location 1 to the outlet at location 2. It hugs the pipe wall, but is shown slightly separated from it as is customary, so that the dashed line can be seen.

The steady state mass balance is $\dot{m} = \dot{m}_1 = \dot{m}_2$, which can be rewritten as

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

Therefore, our first task is to calculate the densities at the inlet and the exit. For this, we shall use the perfect gas equation (also known as the ideal gas law). This can be written in the present context as

$$\rho = \frac{p}{R_g / M_w \, T} = \frac{p \, M_w}{R_g T}$$

where, p is the prevailing pressure, R_g is the universal gas constant, M_w is the molecular weight of the gas, and T is the absolute temperature.

At location 1, $p_1 = 820 \frac{lb_f}{in^2}$ and the absolute temperature is $T_1 = 70^\circ F + 459.7 = 530^\circ R$, when rounded to three significant figures. The value of the universal gas constant is given in the problem statement as $R_g = 10.73 \frac{\left(lb_f / in^2\right) \bullet ft^3}{\left(lb\ mole\right) \bullet {}^\circ R}$, and the molecular weight of natural gas is given as $16 \frac{lb}{lb\ mole}$. Using all this information, we find the density at location 1 to be

$$\rho_1 = \frac{p_1 \, M_w}{R_g T_1} = \frac{820\left(lb_f / in^2\right) \times 16\left(lb / lb\ mole\right)}{10.73 \frac{\left(lb_f / in^2\right) \bullet \left(ft^3\right)}{\left(lb\ mole\right) \bullet \left({}^\circ R\right)} \times 530\left({}^\circ R\right)} = \boxed{2.31 \frac{lb}{ft^3}}$$

At location 2, the pressure is $p_2 = 450 \frac{lb_f}{in^2}$ and the temperature is $T_2 = 58^\circ F + 459.7 = 518^\circ R$. Therefore, we can find the density ρ_2 using

$$\rho_2 = \rho_1 \frac{T_1}{T_2} \frac{p_2}{p_1} = 2.31 \left(\frac{lb}{ft^3}\right) \times \frac{530^\circ R}{518^\circ R} \times \frac{450\left(lb_f / in^2\right)}{820\left(lb_f / in^2\right)} = \boxed{1.30 \frac{lb}{ft^3}}$$

The two areas can be found from the diameters.

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times (2.00\ ft)^2 = \boxed{3.14\ ft^2}$$

$$A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} \times (2.50\ ft)^2 = \boxed{4.91\ ft^2}$$

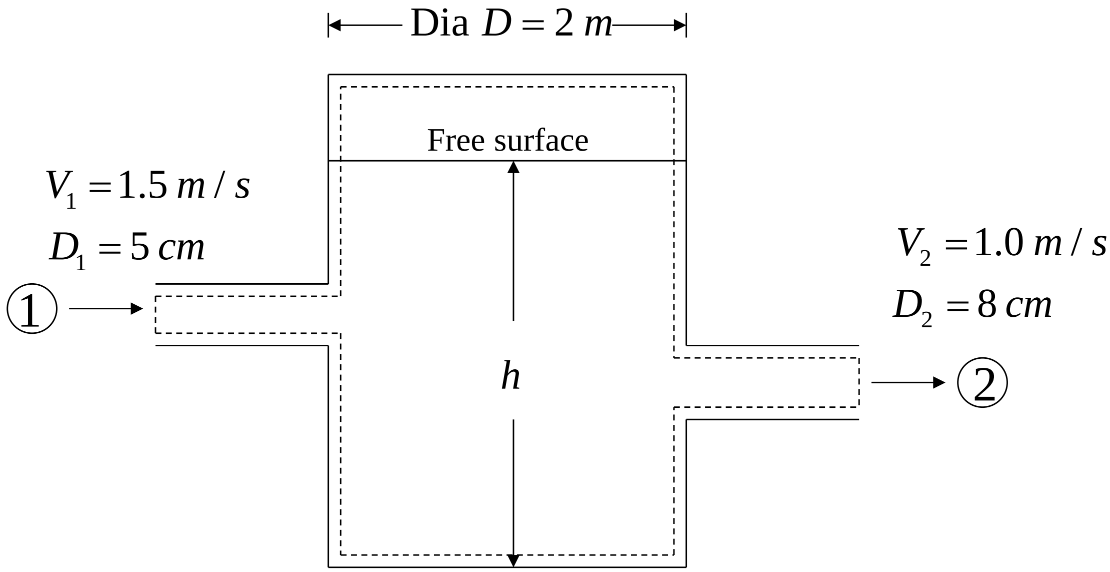
Let us find the mass flow rate.

$$\dot{m}_2 = \dot{m} = \dot{m}_1 = \rho_1 V_1 A_1 = 2.31 \left(\frac{lb}{ft^3} \right) \times 33.7 \left(\frac{ft}{s} \right) \times 3.14 \left(ft^2 \right) = \boxed{244 \frac{lb}{s}}$$

Now, we can calculate the velocity V_2 from

$$V_2 = \frac{\dot{m}_2}{\rho_2 A_2} = \frac{244\ (lb/s)}{1.30\ (lb/ft^3) \times 4.91\ (ft^2)} = \boxed{38.2 \frac{ft}{s}}$$

Example 2 – Unsteady Mass Balance



Is the liquid level in the above tank rising or falling? How fast?

To solve this problem, we must use the unsteady version of the conservation of mass equation applied to a control volume that is shown in the sketch using a dashed line. The control volume occupies the entire interior of the tank, including the inlet and exit pipes. There is an entrance to the control volume at location 1 and an exit at location 2. Liquid flows into the tank at location 1 and flows out at location 2, as shown in the sketch. We begin with

$$\frac{dM}{dt} = -\int_{CS} \rho \left(\boldsymbol{V} \bullet \boldsymbol{n}\right) dA$$

where $M(t)$ is the mass of the liquid in the control volume, which depends on time t. The density of the liquid ρ can be assumed to be constant.

We can write the mass content in the control volume as the sum of the mass of the liquid in the tank, given by $(\pi/4)\rho D^2 h(t)$, and the constant mass in the inlet and outlet pipe sections that are contained the control volume, termed C. The integral on the right side of the unsteady mass conservation equation works out to $\rho(Q_1 - Q_2)$. Using this information, the unsteady mass conservation equation becomes

$$\frac{d}{dt}\left(\frac{\pi}{4}\rho D^2 h + C\right) = \frac{\pi}{4}\rho D^2 \frac{dh}{dt} = \rho\left(Q_1 - Q_2\right) \quad \text{or}$$

$$\boxed{\frac{dh}{dt} = \frac{4\left(Q_1 - Q_2\right)}{\pi D^2}}$$

Therefore, we must calculate the numerical value of the right side in the above equation. The areas of the inlet and outlet can be calculated as follows.

$$A_1 = \frac{\pi}{4}D_1^2 = \frac{\pi}{4}\times\left(0.05\,m\right)^2 = \boxed{1.96\times10^{-3}\ m^2}$$

$$A_2 = \frac{\pi}{4}D_2^2 = \frac{\pi}{4}\times\left(0.08\,m\right)^2 = \boxed{5.03\times10^{-3}\ m^2}$$

Therefore, the volumetric flow rates are

$$Q_1 = V_1 A_1 = 1.5\left(\frac{m}{s}\right)\times 1.96\times10^{-3}\left(m^2\right) = \boxed{2.94\times10^{-3}\ \frac{m^3}{s}}$$

$$Q_2 = V_2 A_2 = 1.0\left(\frac{m}{s}\right)\times 5.03\times10^{-3}\left(m^2\right) = \boxed{5.03\times10^{-3}\ \frac{m^3}{s}}$$

Substitute the above information in the differential equation for the height of the liquid in the tank.

$$\frac{dh}{dt} = \frac{4\left(Q_1 - Q_2\right)}{\pi D^2} = \frac{4\left(2.94\times10^{-3} - 5.03\times10^{-3}\right)m^3/s}{\pi\left(2.0\,m\right)^2} = \boxed{-6.66\times10^{-4}\ \frac{m}{s}}$$

Because the rate of change of the height of liquid in the tank is negative, the liquid level is falling.