# Conduction in the Cylindrical Geometry 

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Chemical engineers encounter conduction in the cylindrical geometry when they analyze heat loss through pipe walls, heat transfer in double-pipe or shell-and-tube heat exchangers, heat transfer from nuclear fuel rods, and other similar situations. Unlike conduction in the rectangular geometry that we have considered so far, the key difference is that the area for heat flow changes from one radial location to another in the cylindrical geometry. This affects the temperature profile in steady conduction. As an example, recall that the steady temperature profile for onedimensional conduction in a rectangular slab is a straight line, provided the thermal conductivity is a constant. In the cylindrical geometry, we find the steady temperature profile to be logarithmic in the radial coordinate in an analogous situation. To see why, let us construct a model of steady conduction in the radial direction through a cylindrical pipe wall when the inner and outer surfaces are maintained at two different temperatures.

## Steady Conduction Through a Straight Cylindrical Pipe Wall

Consider a straight circular pipe of inner radius $r_{1}$, outer radius $r_{2}$ and length $L$ depicted below.
$T_{2}$


Let the steady temperature of the inner surface be $T_{1}$ and that of the outer surface be $T_{2}$, as shown on the sketch. The temperature varies only in the radial direction. For modeling heat flow through the pipe wall, it is convenient to use the end view shown on the following page.


We use a shell balance approach. Consider a cylindrical shell of inner radius $r$ and outer radius $r+\Delta r$ located within the pipe wall as shown in the sketch. The shell extends the entire length $L$ of the pipe. Let $Q(r)$ be the radial heat flow rate at the radial location $r$ within the pipe wall. Then, in the end view shown above, the heat flow rate into the cylindrical shell is $Q(r)$, while the heat flow rate out of the cylindrical shell is $Q(r+\Delta r)$. At steady state, $Q(r)=Q(r+\Delta r)$. Rearrange this result after division by $\Delta r$ as shown below.

$$
\frac{Q(r+\Delta r)-Q(r)}{\Delta r}=0
$$

Now take the limit as $\Delta r \rightarrow 0$. This leads to the simple differential equation

$$
\frac{d Q}{d r}=0
$$

Integration is straightforward, and leads to the result
$Q=$ constant independent of radial location.
The heat flow rate $Q=q_{r} A$, where $A$ is the area of the cylindrical surface normal to the $r-$ direction, and $q_{r}$ is the heat flux in the radial direction. The area of the cylindrical surface is $A=2 \pi r L$, where $L$ is the length of the pipe. From Fourier's law, $q_{r}=-k \frac{d T}{d r}$. Therefore, we find that
$-A k \frac{d T}{d r}=$ constant
Substituting for the area $A$, we can write $-2 \pi r L k \frac{d T}{d r}=$ constant. Because $2 \pi L k$ is constant, this leads to a simple differential equation for the temperature distribution in the pipe wall.
$r \frac{d T}{d r}=C_{1}$
Here, $C_{1}$ is a constant that needs to be determined later. Rearrange this equation as
$d T=C_{1} \frac{d r}{r}$
and integrate both sides to yield

$$
T=C_{1} \ln r+C_{2}
$$

where $C_{2}$ is another constant that needs to be determined. Thus, we find that the steady state temperature distribution in the pipe wall for constant thermal conductivity is logarithmic, in contrast to conduction through a rectangular slab in which case, the steady temperature distribution was found to be linear.

To find the two constants in the solution, we must use boundary conditions on the temperature distribution. The temperature is specified at both the inner and outer pipe wall surfaces. Thus, we can write the boundary conditions as follows.
$T\left(r_{1}\right)=T_{1} \quad T\left(r_{2}\right)=T_{2}$
By substituting these two boundary conditions in the solution for the temperature field in turn, we obtain two equations for the undetermined constants $C_{1}$ and $C_{2}$.
$T_{1}=C_{1} \ln r_{1}+C_{2}$
$T_{2}=C_{1} \ln r_{2}+C_{2}$
These two linear equations can be solved for the values of the constants $C_{1}$ and $C_{2}$ in a straightforward manner. We find

$$
C_{1}=\frac{T_{1}-T_{2}}{\ln \left(r_{1} / r_{2}\right)} \quad C_{2}=T_{1}-\frac{T_{1}-T_{2}}{\ln \left(r_{1} / r_{2}\right)} \ln r_{1}
$$

Substituting the results for the two constants in the solution for the temperature profile, followed by rearrangement, yields the following result for the temperature distribution in the pipe wall.

$$
\frac{T_{1}-T}{T_{1}-T_{2}}=\frac{\ln \left(r / r_{1}\right)}{\ln \left(r_{2} / r_{1}\right)}
$$

A sample sketch of the steady temperature profile in the pipe wall is shown below.


We are ready to evaluate the heat flow rate $Q$.
$Q=q_{r} A=\left(-k \frac{d T}{d r}\right) 2 \pi r L$
Recall that $d T / d r=C_{1} / r$. Substituting in the result for $Q$ leads to
$Q=\left(-k C_{1}\right) 2 \pi L=-2 \pi L k \frac{T_{1}-T_{2}}{\ln \left(r_{1} / r_{2}\right)}=2 \pi L k \frac{T_{1}-T_{2}}{\ln \left(r_{2} / r_{1}\right)}$
This result is a bit hard to remember. Let us try recasting it in a form similar to that we used for steady conduction in a rectangular slab. Let us write
$Q=\frac{\Delta T}{R}$ where $\Delta T=T_{1}-T_{2}$ is the driving force, and $R$ is the resistance to heat flow. We find that the resistance is $R=\ln \left(r_{2} / r_{1}\right) /(2 \pi L k)$. Let us rearrange this result.
$R=\frac{\ln \left(r_{2} / r_{1}\right)}{2 \pi L k}=\frac{r_{2}-r_{1}}{r_{2}-r_{1}} \times \frac{\ln \left(r_{2} / r_{1}\right)}{2 \pi L k}=\frac{r_{2}-r_{1}}{\left(\frac{k\left[2 \pi L r_{2}-2 \pi L r_{1}\right]}{\ln \left(2 \pi L r_{2} / 2 \pi L r_{1}\right)}\right)}=\frac{r_{2}-r_{1}}{k A_{\text {tm }}}$
where we have introduced a new symbol $A_{l m}$, which stands for log mean area. It is defined as $A_{t m}=\frac{A_{2}-A_{1}}{\ln \left(A_{2} / A_{1}\right)}$ where $A_{1}=2 \pi L r_{1}$ is the inner surface area of the pipe wall, and $A_{2}=2 \pi L r_{2}$ is the outer surface area of the pipe wall. The log mean always lies between the two values being averaged. Try comparing it with the more common arithmetic average, which is $\frac{A_{1}+A_{2}}{2}$. You will find that the $\log$ mean is always smaller than the arithmetic average. For $\left(A_{2} / A_{1}\right) \leq 2$, the difference between the two averages is less than $4 \%$.

## Steady Conduction Through Multiple Layers in the Cylindrical Geometry

It is straightforward to extend our analysis of steady state conduction in a pipe wall to multiple layers in the cylindrical geometry. Consider, for example, a pipe of length $L$ carrying hot or cold fluid that needs to be insulated from the surroundings. We add an insulation layer to the outside of the pipe. Here is an end view of the setup.


Air

In the above sketch, region A is the pipe wall, and region B is the insulation layer. The pipe is surrounded by air in this example. Based on our earlier analysis, we can immediately write the steady state heat flow rate $Q$ from the interior wall of the pipe to the outside surface of the insulation in terms of the driving force for conduction in each of the two layers and the resistance of each layer.
$Q=\frac{T_{1}-T_{2}}{R_{\text {pipe }}}=\frac{T_{2}-T_{3}}{R_{\text {ins }}}$ where $R_{\text {pipe }}=\frac{r_{2}-r_{1}}{k_{\text {pipe }} A_{\text {lm, pipe }}}$ and $R_{\text {ins }}=\frac{r_{3}-r_{2}}{k_{\text {ins }} A_{\text {tm, ins }}}$ are the resistances to heat flow in the pipe wall and the insulation layer, respectively. The two log mean areas are defined as follows.
$A_{\text {lm, pipe }}=\frac{A_{2}-A_{1}}{\ln \left(A_{2} / A_{1}\right)}$ and $A_{\text {lm, ins }}=\frac{A_{3}-A_{2}}{\ln \left(A_{3} / A_{2}\right)}$, and $A_{1}=2 \pi L r_{1}, A_{2}=2 \pi L r_{2}, A_{3}=2 \pi L r_{3}$.
We can combine the resistances and use an overall driving force, just as we did in the case of steady conduction through a composite slab. Because
$T_{1}-T_{2}=Q R_{\text {pipe }} \quad T_{2}-T_{3}=Q R_{\text {ins }}$,
adding the two left sides, we obtain
$T_{1}-T_{3}=Q\left(R_{\text {pipe }}+R_{\text {ins }}\right)$
so that we can write

$$
Q=\frac{T_{1}-T_{3}}{\left(R_{\text {pipe }}+R_{\text {ins }}\right)}
$$

It is possible to include additional convective resistances to heat transfer in the interior of the pipe and from the outside surface to the air surrounding the insulated pipe. If we define a heat transfer coefficient $h_{i}$ to describe convective heat transfer between the fluid flowing through the pipe at a temperature $T_{i}$ and the pipe wall, whose interior surface is at a temperature $T_{1}$, we can write
$Q=h_{i} A_{i}\left(T_{i}-T_{1}\right)=\frac{T_{i}-T_{1}}{1 /\left(h_{i} A_{i}\right)}$
where $A_{i}=2 \pi L r_{1}$ is the inside area for heat transfer, and is the same as $A_{1}$ defined earlier. Thus, the convective heat transfer resistance on the inside of the pipe is $1 /\left(h_{i} A_{i}\right)$. Likewise, if the convective heat transfer coefficient between the outside surface of the insulated pipe and the surrounding air is $h_{o}$, and the temperature of the air is $T_{o}$, we can write
$Q=h_{o} A_{o}\left(T_{3}-T_{o}\right)=\frac{T_{3}-T_{o}}{1 /\left(h_{o} A_{o}\right)}$
where $A_{o}=2 \pi L r_{3}$ is the outside surface area for heat transfer, and is the same as $A_{3}$. We see that the convective heat transfer resistance on the outside of the pipe is $1 /\left(h_{o} A_{o}\right)$.

Now, we can write the following result for the heat flow rate.
$Q=\frac{T_{i}-T_{1}}{1 /\left(h_{i} A_{i}\right)}=\frac{T_{1}-T_{2}}{R_{\text {pipe }}}=\frac{T_{2}-T_{3}}{R_{\text {ins }}}=\frac{T_{3}-T_{o}}{1 /\left(h_{o} A_{o}\right)}=U_{i} A_{i}\left(T_{i}-T_{o}\right)=U_{o} A_{o}\left(T_{i}-T_{o}\right)$
where $U_{i}$ is the overall heat transfer coefficient based on the inside heat transfer area, and $U_{o}$ is the overall heat transfer coefficient based on the outside area. Note that in the cylindrical geometry, we have to specify the area upon which the definition of the overall heat transfer coefficient is based, unlike in the rectangular geometry where the area for heat flow did not change across the path. It is straightforward to see that the overall heat transfer coefficients can be obtained from the following result.

$$
\frac{1}{U_{i} A_{i}}=\frac{1}{U_{o} A_{o}}=\sum_{k} R_{k}=\frac{1}{h_{i} A_{i}}+\frac{r_{2}-r_{1}}{k_{\text {pipe }} A_{\text {lm, pipe }}}+\frac{r_{3}-r_{2}}{k_{\text {ins }} A_{\text {lm, ins }}}+\frac{1}{h_{o} A_{o}}
$$

This equation states that the overall resistance to heat transfer, signified by either $1 /\left(U_{i} A_{i}\right)$ or $1 /\left(U_{o} A_{o}\right)$ is comprised of contributions from each individual resistance to heat transfer in series. Other resistances can be added as needed, for example when there is a thermal contact resistance between the pipe wall and the insulation layer, or when fouling occurs in an industrial heat transfer situation. You'll find additional information in your textbook. Also, two good references are the texts by Mills (1) and Holman (2).

So far, we have assumed the thermal conductivity to be a constant. If the temperature variation across a layer is sufficiently large that the thermal conductivity changes significantly, then the simplest approach is to use the thermal conductivity evaluated at the arithmetic average of the temperatures at the inside and outside surfaces of that layer. In problems involving conduction through an insulated pipe wall, the two surface temperatures of the insulation layer may not be known initially. In such cases, you can begin by making an educated guess of the temperatures involved and estimate the thermal conductivity. After calculations are made using this guessed thermal conductivity, the temperatures of the two surfaces of the insulation layer can be calculated. These temperatures are then used in a second iteration to make a more accurate estimate of the average temperature of the insulation layer, and the thermal conductivity at that temperature. Usually, one or two iterations of this type will suffice, and the thermal conductivity of the layer will no longer change appreciably at the next iteration.

## Steady Conduction with Heat Generation in a Cylinder

When a current passes through an electrical wire, because of the resistance to the flow of electricity in the wire, heat is generated. The heat generation can be characterized as a volumetric source of heat $S$. In SI units, $S$ would be measured in $W / \mathrm{m}^{3}$. At steady state, this heat must flow radially out of the wire into the surroundings either directly or through an insulation layer. Likewise, in nuclear fuel rods, a nuclear process within the rod leads to a source of heat.

Another example is the heating of food in a microwave oven, wherein the microwave radiation absorbed by the food leads to heating within it. In all these cases, the center of the wire or fuel rod or the food item would be hot, and the surface would be cooler, because of the need for heat to flow out of the object and into the surroundings. It is useful to be able to estimate the temperature difference between the center and the surface, and the difference in temperature between the surface and the surrounding fluid, typically air, at steady state. Therefore, next we analyze steady radial heat conduction in a cylinder containing a uniform volumetric heat source. A similar analysis can be carried out for a sphere, but is omitted here.

Consider a cylinder of length L and radius $R$ containing a uniform volumetric heat source $S$, and surrounded by air at a temperature $T_{o}$, as shown in the sketch below.

Air at $T_{o}$
$T_{s}$


The cylinder is assumed to be long with insulated ends, so that heat transfer through the cylinder and to the air occurs only in the radial direction. Let the heat transfer coefficient between the surface of the cylinder and the surrounding air be $h_{o}$, and the steady state temperature at the surface of the cylinder be $T_{s}$.

First, we shall use an overall energy balance at steady state to establish the relationship between the surface temperature $T_{s}$ and the surrounding air temperature $T_{o}$. At steady state, all the heat generated within the cylinder must be transferred to the surrounding air. Thus, the rate of heat generation within the cylinder, which is the product of the volume of the cylinder and the heat generation rate per unit volume $S$, must be equal to the heat flux through convective heat transfer to the surrounding air multiplied by the surface area of the cylinder.

$$
Q=\pi R^{2} L S=h_{o}\left(T_{s}-T_{o}\right) 2 \pi R L
$$

Therefore,
$T_{s}-T_{o}=\frac{\pi R^{2} L S}{2 \pi R L h_{o}}=\frac{R}{2 h_{o}} S$ or $T_{s}=T_{o}+\frac{R}{2 h_{o}} S$
Now, let us investigate the temperature distribution within the cylinder at steady state. For this purpose, we begin with a sketch of a differential section of the cylinder located at a radius $r$ and of width $\Delta r$, and extending the entire length $L$ of the cylinder, as shown in the sketch.


The steady state energy balance for the cylindrical shell shown in the sketch can be written as follows.

$$
Q(r)+2 \pi r \Delta r L S=Q(r+\Delta r)
$$

In words, the heat flowing out of the shell at $r+\Delta r$ is the sum of the heat flowing in at the radial location $r$ and the heat generated within the volume of the shell. Rewrite the above equation as

$$
Q(r+\Delta r)-Q(r)=2 \pi r \Delta r L S
$$

Now, divide through by $\Delta r$, and take the limit as $\Delta r \rightarrow 0$.

$$
\operatorname{Limit}_{\Delta r \rightarrow 0}\left(\frac{Q(r+\Delta r)-Q(r)}{\Delta r}\right)=2 \pi r L S
$$

This results in the following ordinary differential equation.

$$
\frac{d Q}{d r}=2 \pi r L S
$$

The heat flow rate $Q=q_{r} A=q_{r} 2 \pi r L$, and $q_{r}$ is the heat flux in the radial direction. From Fourier's law, $q_{r}=-k(d T / d r)$, and using this result yields $Q=-2 \pi r L k(d T / d r)$. Substitute for $Q$ in the differential equation, treating the thermal conductivity $k$ as a constant, and rearrange to obtain the following second order ordinary differential equation for the steady temperature distribution in the cylinder.
$\frac{1}{r} \frac{d}{d r}\left(r \frac{d T}{d r}\right)=-\frac{S}{k}$

It is straightforward to integrate this equation if we keep it intact without simplifying the differentiations using the product rule. First, we rewrite it as
$\frac{d}{d r}\left(r \frac{d T}{d r}\right)=-\frac{S}{k} r$
which can be integrated immediately, because the left side is just the derivative of $r \frac{d T}{d r}$ with respect to $r$. Thus, integration leads to

$$
r \frac{d T}{d r}=-\frac{S}{2 k} r^{2}+C_{1}
$$

where we have introduced an arbitrary constant of integration $C_{1}$. Rearrange this equation as

$$
\frac{d T}{d r}=-\frac{S}{2 k} r+\frac{C_{1}}{r}
$$

and it can be integrated once again! The result for the temperature distribution in the rod is
$T(r)=-\frac{S}{4 k} r^{2}+C_{1} \ln r+C_{2}$
where a second arbitrary constant of integration has been introduced. Clearly, the temperature has to be finite everywhere, including at $r=0$. To satisfy this constraint, we can see that the constant $C_{1}$ must be zero. Thus,

$$
T(r)=-\frac{S}{4 k} r^{2}+C_{2}
$$

To find the remaining arbitrary constant, we use the boundary condition that the temperature at the surface $r=R$ is $T_{s}$. This condition is written formally as follows.
$T(R)=T_{s}$
Evaluating both sides of the result for the temperature field at the surface $r=R$,
$T_{s}=-\frac{S}{4 k} R^{2}+C_{2}$
so that $C_{2}=T_{s}+\frac{S}{4 k} R^{2}$. Substitute this result for $C_{2}$ in the temperature field to obtain
$T=T_{s}+\frac{S}{4 k}\left(R^{2}-r^{2}\right)$
Can you guess where the temperature will be a maximum? From the solution, we can see that this will occur when $r=0$, which is the axis of the rod. This maximum value is found to be
$T_{\max }=T(0)=T_{s}+\frac{S}{4 k} R^{2}$

A sample sketch of the radial temperature distribution in the rod is given on the following page.


## References

1. A.F. Mills, Heat Transfer, Second Edition, Prentice-Hall, New Jersey, 1999.
2. J.P. Holman, Heat Transfer, Tenth Edition, McGraw-Hill, New York, 2010.
