AE/ME 455 Formula Page for Exam II

Simple harmonic function

$$z(t) = a\cos\omega t + b\sin\omega t = C\cos(\omega t - \psi) = C\sin(\omega t + \psi_o)$$

$$C = \sqrt{a^2 + b^2}; \qquad \psi = \tan^{-1}\left(\frac{b}{a}\right); \qquad \psi_o = \tan^{-1}\left(\frac{a}{b}\right)$$

Harmonically excited single degree of freedom system

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = \omega_n^2\delta\sin\omega t$$

$$x_{ss}(t) = X_{ss}\sin(\omega t - \phi)$$

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = \omega_n^2\delta\cos\omega t$$

$$x_{ss}(t) = X_{ss}\cos(\omega t - \phi)$$

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = \omega_n^2\delta\cos(\omega t - \beta)$$

$$x_{ss}(t) = X_{ss}\cos(\omega t - \beta - \phi)$$

$$X_{ss} = \frac{\delta}{\sqrt{(1-r^2)^2 + (2\varsigma r)^2}}; \qquad \phi = \tan^{-1}\left(\frac{2\varsigma r}{1-r^2}\right); \qquad r = \frac{\omega}{\omega_n}$$

Harmonically excited single degree of freedom system - base excitation

Base excitation has form $\underline{\sin\omega t}$:

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 Y[2\varsigma r\cos\omega t + \sin\omega t] = \omega_n^2\overline{Y}\cos(\omega t - \alpha)$$

$$\overline{Y} = Y\sqrt{1 + (2\varsigma r)^2}; \qquad \alpha = \tan^{-1}\left(\frac{1}{2\varsigma r}\right)$$

$$x_{ss}(t) = Q_{ss}\cos(\omega t - \alpha - \phi)$$

$$Q_{ss} = \frac{\overline{Y}}{\sqrt{(1-r^2)^2 + (2\varsigma r)^2}}; \qquad \phi = \tan^{-1}\left(\frac{2\varsigma r}{1-r^2}\right); \qquad r = \frac{\omega}{\omega_n}$$

Base excitation has form $\cos\omega t$:

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 Y[\cos\omega t - 2\varsigma r\sin\omega t] = \omega_n^2\overline{Y}\sin(\omega t - \alpha)$$

$$\overline{Y} = Y\sqrt{1+(2\varsigma r)^2}; \qquad \alpha = \tan^{-1}\left(\frac{1}{2\varsigma r}\right)$$

$$x_{ss}(t) = Q_{ss}\sin(\omega t - \alpha - \phi)$$

$$Q_{ss} = \frac{\overline{Y}}{\sqrt{(1-r^2)^2 + (2\varsigma r)^2}}; \qquad \phi = \tan^{-1}\left(\frac{2\varsigma r}{1-r^2}\right); \qquad r = \frac{\omega}{\omega_n}$$

Periodic excitation represented by Fourier series

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = \omega_n^2\delta[\frac{a_o}{2} + \sum_{j=1}^{\infty} a_j\cos(j\omega t) + \sum_{j-1}^{\infty} b_j\sin(j\omega t)]$$

$$x_{ss}(t) = \delta\left[\frac{a_o}{2} + \sum_{j=1}^{\infty}\frac{a_j}{\sqrt{(1-j^2r^2)^2 + (2\varsigma jr)^2}}\cos(j\omega t - \phi_j) + \sum_{j=1}^{\infty}\frac{b_j}{\sqrt{(1-j^2r^2)^2 + (2\varsigma jr)^2}}\sin(j\omega t - \phi_j)\right]$$

$$r = \frac{\omega}{\omega_n}; \qquad jr = \frac{j\omega}{\omega_n}; \qquad \phi_j = \tan^{-1}\left(\frac{2\varsigma jr}{1-j^2r^2}\right)$$

Fourier Series

$F(t)$ periodic with period = τ; fundamental frequency $= \omega = \dfrac{2\pi}{\tau}$

Fourier series for $F(t)$ is:

$$F(t) = \frac{a_o}{2} + \sum_{j=1}^{\infty} a_j\cos(j\omega t) + \sum_{j=1}^{\infty} b_j\sin(j\omega t)$$

$$a_o = \frac{\omega}{\pi}\int_0^{2\pi/\omega} F(t)dt; \qquad a_j = \frac{\omega}{\pi}\int_0^{2\pi/\omega} F(t)\cos(j\omega t)dt; \qquad b_j = \frac{\omega}{\pi}\int_0^{2\pi/\omega} F(t)\sin(j\omega t)dt$$