Clarkson University
Department of Mechanical and Aeronautical Engineering
AE/ME 455 Mechanical Vibrations
Spring 2005 – Exam II
Thursday, March 31, 2005

Name: SOLUTIONS

Student No.: ____________________

Instructions: Read the problems carefully and space your time so as to gain the maximum number of points of credit. You will be given partial credit only for those steps in a solution that are leading to a logical conclusion. Please write neatly and clearly and make clear unambiguous sketches; cross out any work you do not wish to be considered. This exam has 3 main problems with multiple parts worth 80 points total. The exam is CLOSED BOOK and CLOSED NOTES. You have 75 minutes to complete the exam.

Problem 1: (35 points)
Vibration of the system shown below is excited by the application of an harmonic displacement to the right hand node (point B) of spring k_1 as shown.

$m = 10\text{ kg}$

$J_o = \frac{mr^2}{2}$

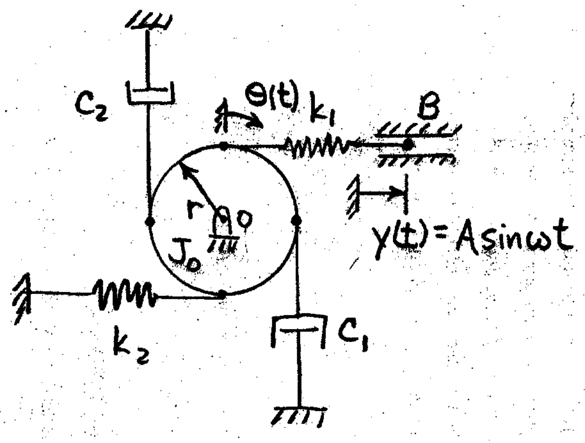
$r = 1\text{ m}$

$c_1 = c_2 = 25\text{N}\cdot\text{s/m}$

$k_1 = k_2 = 250\text{N/m}$

$A = 0.1\text{m}$

$\omega = 20\text{rad/s}$

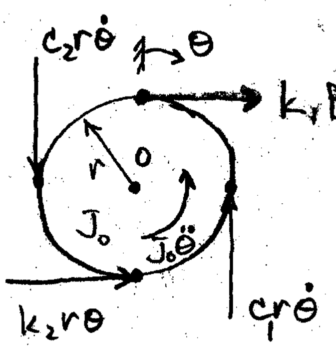


(15 points) A. Determine the equation of motion governing the response $\theta(t)$ of the system.

(10 points) B. Compute the steady state amplitude of the response.

(5 points) C. Compute the phase angle between the steady state response and the excitation.

(5 points) D. Compute the steady state amplitude of the force exerted by the damping element c_1 on the rotating mass J_o.



$$-J_o\ddot{\theta} - (c_1+c_2)r\dot{\theta}r + k_1[y(t) - r\theta]r - k_2 r\theta r = 0$$

$$\frac{mr^2}{2}\ddot{\theta} + (c_1+c_2)r^2\dot{\theta} + (k_1+k_2)r^2\theta = k_1 r y(t)$$

(A.) $$\boxed{\ddot{\theta} + 2\left(\frac{c_1+c_2}{m}\right)\dot{\theta} + 2\left(\frac{k_1+k_2}{m}\right)\theta = \frac{2k_1}{mr}A\sin\omega t}$$

Prob 1 cont-

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \omega_n^2\left(\frac{2k_1}{m\omega_n^2}\frac{A}{r}\right)\sin\omega t$$

$$\omega_n^2 = 2\left(\frac{k_1+k_2}{m}\right) = 2\left(\frac{500\ N/m}{10\ kg}\right) = 100\ rad/s^2$$

$$\omega_n = 10\ rad/s$$

$$\zeta = \frac{2(c_1+c_2)}{2m\omega_n} = \frac{2(50\ N\cdot s/m)}{2(10\ kg)(10\ rad/s)} = 0.5$$

$$\delta = \left(\frac{2k_1}{m\omega_n^2}\frac{A}{r}\right) = \frac{500\ N/m}{10\ kg(100\ rad/s^2)}\ \frac{0.1\ m}{1\ m} = 0.05\ rad$$

$$r = \frac{\omega}{\omega_n} = \frac{20\ rad/s}{10\ rad/s} = 2$$

$$\theta_{ss\,max} = \frac{\delta}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{0.05\ rad}{\sqrt{9+4}}$$

(B.) $\boxed{\theta_{ss\,max} = 0.014\ rad}$

$$\theta_{ss}(t) = \theta_{ss\,max}\sin(\omega t - \phi)$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2}{-3}\right)$$

(C.) $\boxed{\phi = 146.3^\circ = 2.55\ rad}$

Prob 1 cont -

$$F_{c_1,ss} = r c_1 \dot{\theta}_{ss} = r c_1 \theta_{ss_{max}} \omega \cos(\omega t - \phi)$$

$$F_{c_1 ss max} = r c_1 \theta_{ssmax} \omega$$

$$F_{c_1 ssmax} = (1\,m)(25\,N\cdot s/m)(0.014\,rad)(20\,\tfrac{rad}{s})$$

(D.) $\boxed{F_{c_1 ssmax} = 7\,N}$

Problem 2: (30 points)
The vibration of the system shown below is excited by motion of the platform. The platform motion is specified by its acceleration history $\ddot{y}(t) = \lambda \cos \omega t$; the motion of the mass m is denoted by $x(t)$. The relative motion between the platform and the mass is denoted by $z(t) = y(t) - x(t)$.

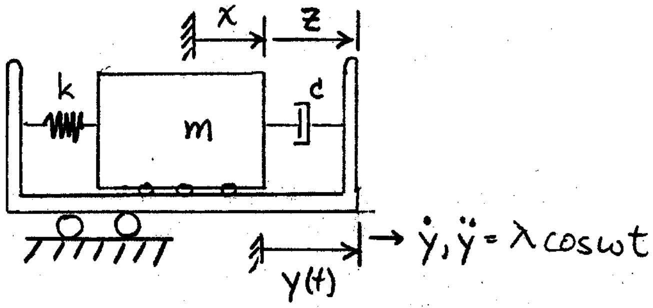
$\lambda = 1\text{m/s}^2$

$k = 100\text{N/m}$

$c = 2\text{N} \cdot \text{s/m}$

$m = 1\text{kg}$

$\omega = 9\text{rad/s}$

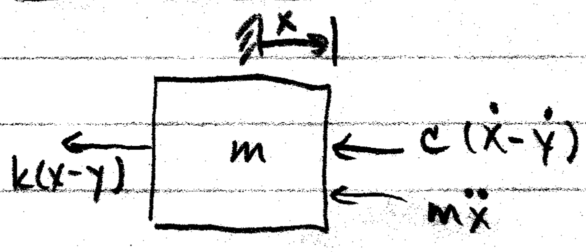


(15 points) A. Determine the equation of motion governing the relative motion $z(t)$.

(8 points) B. Compute the steady state amplitude of the relative motion (i.e. the steady state amplitude of $z(t)$).

(7 points) C. Compute the steady state amplitude of the net force exerted on the mass by the spring and damper combined.

Problem 2



$$-m\ddot{x} - c(\dot{x}-\dot{y}) - k(x-y) = 0$$

$$z = y - x$$

$$m(\ddot{z}-\ddot{y}) + c\dot{z} + kz = 0$$

$$m\ddot{z} + c\dot{z} + kz = m\ddot{y}$$

(A.) $\boxed{m\ddot{z} + c\dot{z} + kz = m\lambda\cos\omega t}$

$$\ddot{z} + \frac{c}{m}\dot{z} + \frac{k}{m}z = \lambda\cos\omega t$$

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \omega_n^2\left(\frac{\lambda}{\omega_n^2}\right)\cos\omega t$$

$$\omega_n^2 = \frac{k}{m} = \frac{100\,\text{N/m}}{1\,\text{kg}} = 100\,\text{rad/s}^2$$

$$\omega_n = 10\,\text{rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{2\,\text{N}\cdot\text{s/m}}{2(1\,\text{kg})(10\,\text{rad/s})} = 0.1$$

$$r = \frac{\omega}{\omega_n} = \frac{9}{10} = 0.9$$

$$\delta = \frac{\lambda}{\omega_n^2} = \frac{1\,\text{m/s}^2}{100\,\text{rad/s}^2} = 0.01\,\text{m}$$

$$Z_{ss\,max} = \frac{\delta}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.01\,\text{m}}{\sqrt{(1-0.9^2)^2 + (0.18)^2}}$$

(B.) $\boxed{Z_{ss\,max} = 0.038\,\text{m}}$

Prob 2 cont - $z_{ss} = z_{ssmax}\cos(\omega t - \phi)$

$$F_{ss} = c(\dot{x} - \dot{y}) + k(x - y) = -m\ddot{x}$$

$$F_{ss} = m(\ddot{z} - \ddot{y})$$

$$= -m z_{ssmax}\,\omega^2\cos(\omega t - \phi) - m\lambda\cos\omega t$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{0.18}{0.19}\right) \approx 45^\circ$$

$$\cos\phi \approx \sqrt{2}/2 \qquad \sin\phi \simeq \sqrt{2}/2$$

$$F_{ss} = \left[-m z_{ssmax}\,\omega^2\frac{\sqrt{2}}{2} - m\lambda\right]\cos\omega t - m z_{ssmax}\,\omega^2\frac{\sqrt{2}}{2}\sin\omega t$$

$$= -3.18\cos\omega t - 2.18\sin\omega t \quad N$$

$$\boxed{F_{ssmax} = \sqrt{(-3.18)^2 + (-2.18)^2} = 3.86\,N}$$

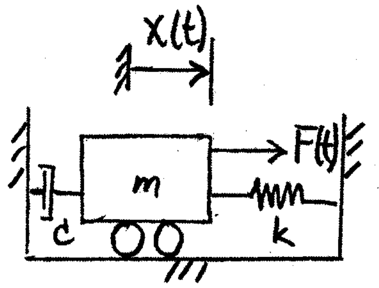
Problem 3: (15 points)

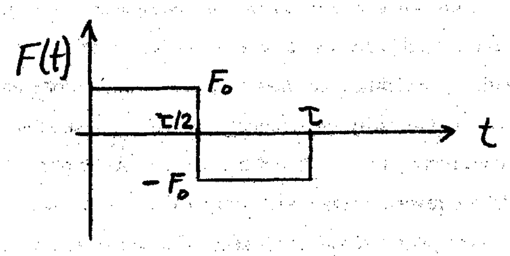
The vibration of the system shown below is excited by the excitation *F(t)*; one cycle of which is plotted to the right.

$k = 4000\text{N/m}$

$c = 2000\text{N}\cdot\text{s/m}$

$m = 1000\text{kg}$





F(t) has a period of τ = 0.2 s, an amplitude of $F_o = 100\text{N}$, and is represented by the following Fourier series:

$$F(t) = \frac{4F_o}{\pi} \sum_{n=1,3,5...}^{\infty} \frac{1}{n} \sin(n\omega t); \quad \omega = \frac{2\pi}{\tau}$$

(5 points) A. Compute the fundamental frequency ω of the excitation.

(10 points) B. If the steady state response of the system to this excitation is

$$x_{ss}(t) = \sum_{n=1,3,5...}^{\infty} X_n \frac{1}{n} \sin(n\omega t - \phi_n)$$

compute X_n for *n* = *1*, i.e. compute X_1.

Problem 3

$$m\ddot{x}(t) + c\dot{x} + kx = F(t)$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2\left(\frac{1}{m\omega_n^2}\right)F(t)$$

$$\omega_n^2 = \frac{k}{m} = \frac{4000\ \text{N/m}}{1000\ \text{kg}} = 4\ \text{rad/s}^2$$

$$\omega_n = 2\ \text{rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{2000\ \text{N}\cdot\text{s/m}}{2(1000\text{kg})(2\text{rad/s})} = 0.5$$

(A.) $$\boxed{\omega = \frac{2\pi}{\tau} = \frac{2\pi}{0.2s} = 31.42\ \text{rad/s}}$$

$$r = \frac{\omega}{\omega_n} = 15.71\ ;\ \delta = \frac{1}{m\omega_n^2} = 2.5\times10^{-4}\ \text{s}^2/\text{kg}$$

$$F(t) = \frac{4F_0}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n}\sin(n\omega t)$$

$$X_{ss}(t) = \delta\frac{4F_0}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1/n}{\sqrt{(1-n^2r^2)^2+(2\zeta nr)^2}}\sin(n\omega t-\phi_n)$$

$$X_n = \frac{4\delta F_0/\pi}{\sqrt{(1-n^2r^2)^2+(2\zeta r)^2}}$$

(B.) $n=1 \Rightarrow$ $$X_1 = \frac{4(2.5\times10^{-4}\,\text{s}^2/\text{kg})(100\text{N})/\pi}{\sqrt{(1-(15.71)^2)^2+(15.71)^2}} = \boxed{1.29\times10^{-4}\ \text{m}}$$

AE/ME 455 Formula Page for Exam II

Simple harmonic function

$$z(t) = a\cos\omega t + b\sin\omega t = C\cos(\omega t - \psi) = C\sin(\omega t + \psi_o)$$

$$C = \sqrt{a^2 + b^2}; \qquad \psi = \tan^{-1}\left(\frac{b}{a}\right); \qquad \psi_o = \tan^{-1}\left(\frac{a}{b}\right)$$

Harmonically excited single degree of freedom system

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = \omega_n^2\delta\sin\omega t$$
$$x_{ss}(t) = X_{ss}\sin(\omega t - \phi)$$

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = \omega_n^2\delta\cos\omega t$$
$$x_{ss}(t) = X_{ss}\cos(\omega t - \phi)$$

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = \omega_n^2\delta\cos(\omega t - \beta)$$
$$x_{ss}(t) = X_{ss}\cos(\omega t - \beta - \phi)$$

$$X_{ss} = \frac{\delta}{\sqrt{(1-r^2)^2 + (2\varsigma r)^2}}; \qquad \phi = \tan^{-1}\left(\frac{2\varsigma r}{1-r^2}\right); \qquad r = \frac{\omega}{\omega_n}$$

Harmonically excited single degree of freedom system - base excitation

Base excitation has form $\sin\omega t$:

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 Y[2\varsigma r\cos\omega t + \sin\omega t] = \omega_n^2\overline{Y}\cos(\omega t - \alpha)$$

$$\overline{Y} = Y\sqrt{1 + (2\varsigma r)^2}; \qquad \alpha = \tan^{-1}\left(\frac{1}{2\varsigma r}\right)$$

$$x_{ss}(t) = Q_{ss}\cos(\omega t - \alpha - \phi)$$

$$Q_{ss} = \frac{\overline{Y}}{\sqrt{(1-r^2)^2 + (2\varsigma r)^2}}; \qquad \phi = \tan^{-1}\left(\frac{2\varsigma r}{1-r^2}\right); \qquad r = \frac{\omega}{\omega_n}$$

Base excitation has form $\cos\omega t$:

$$\ddot{x} + 2\varsigma\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 Y[\cos\omega t - 2\varsigma r \sin\omega t] = \omega_n^2 \overline{Y} \sin(\omega t - \alpha)$$

$$\overline{Y} = Y\sqrt{1 + (2\varsigma r)^2}; \qquad \alpha = \tan^{-1}\left(\frac{1}{2\varsigma r}\right)$$

$$x_{ss}(t) = Q_{ss} \sin(\omega t - \alpha - \phi)$$

$$Q_{ss} = \frac{\overline{Y}}{\sqrt{(1-r^2)^2 + (2\varsigma r)^2}}; \qquad \phi = \tan^{-1}\left(\frac{2\varsigma r}{1-r^2}\right); \qquad r = \frac{\omega}{\omega_n}$$

Periodic excitation represented by Fourier series

$$\ddot{x} + 2\varsigma\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 \delta[\frac{a_o}{2} + \sum_{j=1}^{\infty} a_j \cos(j\omega t) + \sum_{j-1}^{\infty} b_j \sin(j\omega t)]$$

$$x_{ss}(t) = \delta\left[\frac{a_o}{2} + \sum_{j=1}^{\infty} \frac{a_j}{\sqrt{(1-j^2r^2)^2 + (2\varsigma jr)^2}} \cos(j\omega t - \phi_j) + \sum_{j=1}^{\infty} \frac{b_j}{\sqrt{(1-j^2r^2)^2 + (2\varsigma jr)^2}} \sin(j\omega t - \phi_j)\right]$$

$$r = \frac{\omega}{\omega_n}; \qquad jr = \frac{j\omega}{\omega_n}; \qquad \phi_j = \tan^{-1}\left(\frac{2\varsigma jr}{1-j^2r^2}\right)$$

Fourier Series

$F(t)$ periodic with period = τ; fundamental frequency $= \omega = \dfrac{2\pi}{\tau}$

Fourier series for $F(t)$ is:

$$F(t) = \frac{a_o}{2} + \sum_{j=1}^{\infty} a_j \cos(j\omega t) + \sum_{j=1}^{\infty} b_j \sin(j\omega t)$$

$$a_o = \frac{\omega}{\pi}\int_0^{2\pi/\omega} F(t)dt; \qquad a_j = \frac{\omega}{\pi}\int_0^{2\pi/\omega} F(t)\cos(j\omega t)dt; \qquad b_j = \frac{\omega}{\pi}\int_0^{2\pi/\omega} F(t)\sin(j\omega t)dt$$