

Clarkson University  
Department of Mechanical and Aeronautical Engineering  
**AE/ME 455 Mechanical Vibrations**  
Spring 2004 – Final Exam  
Wednesday, April 28, 2004

Name: SOLUTIONS

Student No.: \_\_\_\_\_

**Instructions:** Read the problems carefully and space your time so as to gain the maximum number of points of credit. You will be given partial credit only for those steps in a solution that are leading to a logical conclusion. Please write neatly and clearly and make clear unambiguous sketches; cross out any work you do not wish to be considered. This exam has 4 main problems with multiple parts worth 120 points total. The exam is CLOSED BOOK and CLOSED NOTES. You have 180 minutes to complete the exam.

**Problem 1: (20 points)**

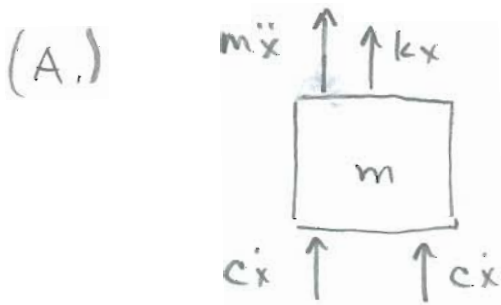
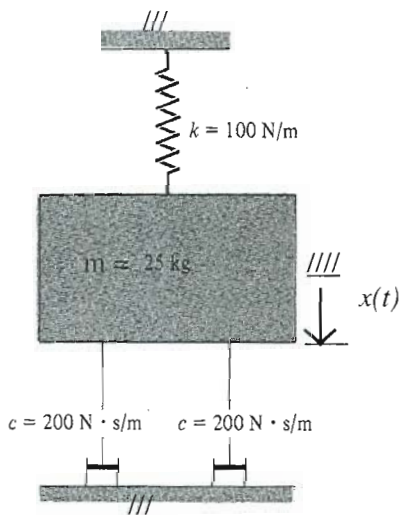
The mechanical system shown below is subjected to the initial excitation:

$$x(0) = 0.01 \text{ m}$$

$$\dot{x}(0) = 0.0$$

where  $x$  is taken relative to the equilibrium position of the 25 kg mass. Compute:

- A. (5 points) The undamped natural frequency.
- B. (5 points) The dimensionless damping ratio.
- C. (5 points) The time required to return 95% of the distance back to the static equilibrium position.
- D. (5 points) The value of the damping coefficient  $c$  that would make the system critically damped.



$$\sum F = -m\ddot{x} - 2cx - kx = 0$$

$$\ddot{x} + \frac{2c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100 \text{ N/m}}{25 \text{ kg}}} = \boxed{2 \text{ rad/s}}$$

(B.)

$$2\zeta\omega_n = \frac{2c}{m} \Rightarrow \zeta = \frac{c}{m\omega_n} = \frac{200 \text{ N·s/m}}{25 \text{ kg} (2 \text{ rad/s})}$$

$$\boxed{\zeta = 4}$$

PROB 1 CONT -  $\zeta > 1$  so system is OVERDAMPED

$$(c.) \Rightarrow x(t) = [C_1 e^{+\omega_d t} + C_2 e^{-\omega_d t}] e^{-\zeta \omega_n t}$$

$$\omega_d = (\sqrt{\zeta^2 - 1}) \omega_n = \sqrt{15} \text{ (2 rad/s)} = 7.746 \text{ rad/s}$$

$$x(0) = C_1 + C_2 = 0.01 \text{ m}$$

$$\dot{x}(0) = \omega_d (C_1 - C_2) - \zeta \omega_n (C_1 + C_2) = 0$$

$$\begin{bmatrix} 1 & 1 \\ \omega_d - \zeta \omega_n & -\omega_d - \zeta \omega_n \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0.01 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1.254 & -15.746 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0.01 \\ 0 \end{Bmatrix}$$

$$C_1 = \frac{0.01(-15.746)}{-15.492} = 0.0102 \text{ m}$$

$$C_2 = \frac{-0.01(-1.254)}{-15.492} = -0.0002 \text{ m}$$

$$\Rightarrow x(t) = e^{-8t} [0.0102 e^{7.746t} - 0.0002 e^{-7.746t}] \text{ m}$$

95% back from  $x(0) = 0.01 \text{ m} \Rightarrow x = 0.0005 \text{ m}$

$$0.0005 = 0.0102 e^{-1.254t} - 0.0002 e^{-15.746t}$$

$$\Rightarrow \boxed{t \approx 12 \text{ s}} \text{ iteratively solved}$$

PROB 1 CONT -

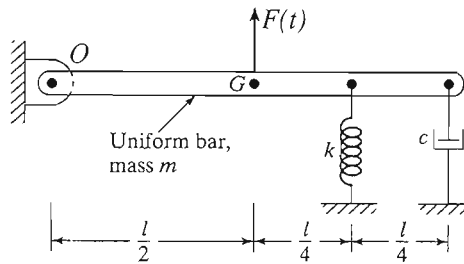
$$(D.) \quad \zeta = \frac{c_{cr}}{m\omega_n} = 1$$

$$\begin{aligned} c_{cr} &= m\omega_n = (25 \text{ kg})(2 \text{ rad/s}) \\ &= \boxed{50 \text{ N}\cdot\text{s/m}} . \end{aligned}$$

**Problem 2: (30 points)**

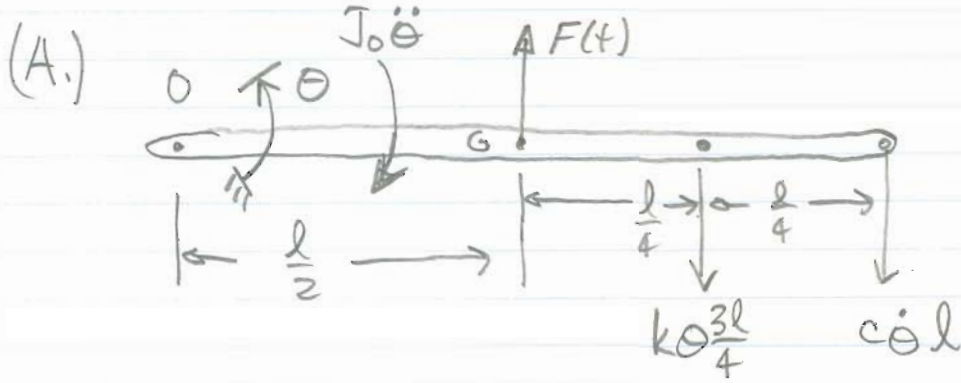
For the system shown below:

- A. (10 points) Write the equation of motion of the system in terms of the system properties given in the figure.
- B. (10 points) Using the following values for the system properties:  $m = 10$  kg,  $k = 5000$  N/m,  $l = 1$  m,  $c = 1000$  N·s/m, determine the steady state response for the rotation of the bar if  $F(t) = 250\sin\omega t$  N where the excitation frequency is  $\omega = 105$  rad/s. Include numerical values of the steady state amplitude and the phase angle.
- C. (10 points) For the same system properties, determine the steady state response of the system (rotation of the bar) if  $F(t)$  is a constant load applied at time  $t=0$  such that the load magnitude is  $F_0 = 250$  N.



**Note:** the mass moment of inertia of a slender bar of length  $l$  and mass  $m$  about an axis parallel to  $O$  passing through the center of mass is  $J_{com} = ml^2/12$ . The parallel axis theorem states that the mass moment of inertia for a parallel axis is  $J_{axis} = J_{com} + mb^2$ , where  $b$  is the distance between the axis passing through the center of mass and the parallel axis.

## PROBLEM 2



$$\sum M = -J_0 \ddot{\theta} - (c \dot{\theta} l) l - k \theta \left(\frac{3l}{4}\right) \frac{3l}{4} + F(t) \frac{l}{2} = 0$$

$$J_0 \ddot{\theta} + c l^2 \dot{\theta} + \frac{9kl^2}{16} \theta = \frac{l}{2} F(t)$$

$$J_0 = \frac{ml^2}{12} + m \left(\frac{l}{2}\right)^2 = \frac{ml^2}{12} (1+3) = \frac{ml^2}{3}$$

$$\Rightarrow \frac{ml^2}{3} \ddot{\theta} + c l^2 \dot{\theta} + \frac{9kl^2}{16} \theta = \frac{l}{2} F(t)$$

$$\text{or } \ddot{\theta} + \frac{3c}{m} \dot{\theta} + \frac{27k}{16m} \theta = \frac{3}{2ml} F(t)$$

$\frac{\text{kg} \cdot \text{m} / \text{s}^2}{\text{kg} \cdot \text{m}}$

$\text{kg} \cdot \text{m}$

PROB 2 CONT -

$$(B.) \quad \ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \omega_n^2\left(\frac{3}{2ml\omega_n^2}\right)F_0\sin\omega t$$

$$\zeta = \frac{3c}{2m\omega_n} \quad \omega_n = \sqrt{\frac{27k}{16m}} = \sqrt{\frac{27(5000N/m)}{16(10kg)}} = 29 \frac{\text{rad}}{\text{s}}$$

$$\zeta = \frac{3(1000N\cdot s/m)}{2(10kg)(29\text{rad/s})} = 5.17$$

$$r = \frac{\omega}{\omega_n} = \frac{105}{29} = 3.62$$

$$\theta_{ss} = \Theta \sin(\omega t - \phi)$$

$$\Theta = \frac{3F_0/2ml\omega_n^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{.044 \text{ rad}}{\sqrt{(1-3.62^2)^2 + (2 \cdot 5.17 \cdot 3.62)^2}}$$

$$\boxed{\Theta = 1.13 \times 10^{-3} \text{ rad}}$$

$$\phi = \tan^{-1}\left[\frac{2\zeta r}{1-r^2}\right] = \tan^{-1}\left[\frac{37.43}{-12.10}\right] = \boxed{2.83 \text{ rad}}$$

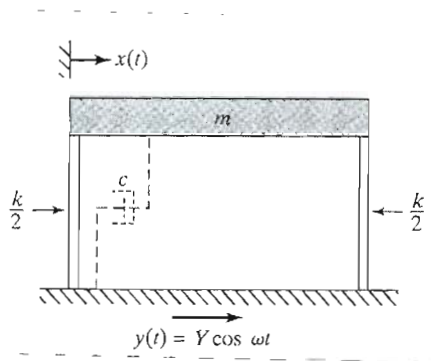
(C.) STEADY STATE  $\Rightarrow \ddot{\theta} = \dot{\theta} = 0$

$$\Rightarrow \theta = \frac{16}{18kl} F_0 = \frac{8(250N)}{9(5000N/m)(1m)} = \boxed{0.044 \text{ rad}}$$

**Problem 3: (30 points)**

In the structure shown below the mass  $m$  is supported by the two columns, each with a bending stiffness of  $k/2$ . Damping of the horizontal motion of the mass is provided by the dashed element shown with viscous damping coefficient  $c$ . The base of the structure undergoes the motion  $y(t)=Y\cos\omega t$ .

- A. (10 points) Write the equation of motion of the system in terms of the properties shown on the diagram.
- B. (10 points) Compute the steady state amplitude of the motion of the mass  $m$  if  $m=4000$  kg,  $k=200000$  N/m,  $\omega=25$  rad/s,  $Y=0.005$  m and  $c=23000$  N·s/m.
- C. (10 points) Compute the amplitude of the total steady state force transmitted to the 4000 kg mass through the two columns.



Problem 3



$$\sum F = -m\ddot{x} - c(\dot{x} - \dot{y}) - 2\frac{k}{2}(x - y) = 0$$

(A.)

$$\left. \begin{aligned} m\ddot{x} + c\dot{x} + kx &= c\dot{y} + ky \\ \text{or } m\ddot{x} + c\dot{x} + kx &= kY\cos\omega t - c\omega Y\sin\omega t \end{aligned} \right\}$$

(B.)

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \ddot{Y}(\omega_n^2\cos\omega t - 2\zeta\omega_n\omega\sin\omega t)$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \omega_n^2\ddot{Y}[\cos\omega t - 2\zeta r\sin\omega t]$$
$$\zeta = \frac{c}{2m\omega_n} \quad \omega_n = \sqrt{\frac{k}{m}} \quad r = \frac{\omega}{\omega_n}$$

$$Q_{ss} = \frac{Y \sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \quad Y = 0.005 \text{ m}$$

$$\omega_n = \sqrt{\frac{200000 \text{ N/m}}{4000 \text{ kg}}} = 7.07 \text{ rad/s} \quad r = \frac{25}{7.07} = 3.54$$

$$\zeta = \frac{23000}{2(4000)(7.07)} = 0.407$$

$$Q_{ss} = \frac{0.005 \sqrt{1 + (2 \cdot 0.407 \cdot 3.54)^2}}{\sqrt{(1 - (3.54)^2)^2 + (2 \cdot 0.407 \cdot 3.54)^2}} = \boxed{1.28 \times 10^{-3} \text{ m}}$$

PROB 3 CONT - (C)  $\vec{b} \rightarrow \boxed{m} \leftarrow m\ddot{x} \quad f = m\ddot{x}$

$$f_{ss} = m\ddot{x}_{ss}$$

$$x_{ss} = Q_{ss} \sin(\omega t - \alpha - \phi)$$

$$\ddot{x}_{ss} = -\omega^2 Q_{ss} \sin(\omega t - \alpha - \phi)$$

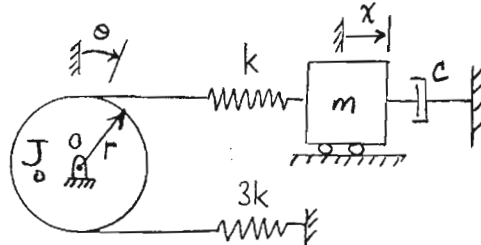
$$\Rightarrow \overline{F}_{ss} = m\omega^2 Q_{ss}$$

$$= (4000 \text{ kg})(25 \text{ rad/s})^2 (1.28 \times 10^{-3} \text{ m})$$

$$F_{ss} = \boxed{3200 \text{ N}}$$

**Problem 4: (40 points)**

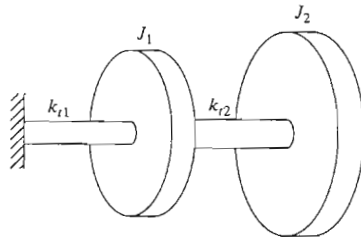
- A. (10 points) In the two degree of freedom mechanical system shown below, the coordinates used to describe the motion are the rotation of the disk  $\theta$ , and the translation of the mass  $x$ . Using the system properties shown on the diagram, derive the equations of motion for the system and express them in matrix form.



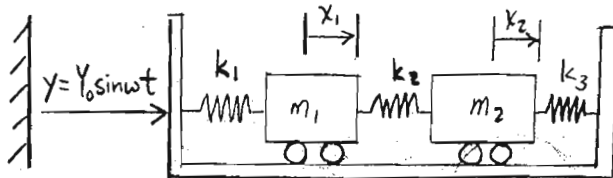
- B. (20 points) In the two degree of freedom mechanical system shown below, the coordinates used to describe the motion are the rotations of the disks,  $\theta_1$  and  $\theta_2$ . The equations of motion for the system are:

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Compute the natural frequencies and normal mode ratios of the system if  $k_{t1} = 100 \text{ N} \cdot \text{m}/\text{rad}$ ,  $k_{t2} = 200 \text{ N} \cdot \text{m}/\text{rad}$ ,  $J_1 = 2.0 \text{ N} \cdot \text{m} \cdot \text{s}^2$ ,  $J_2 = 4.0 \text{ N} \cdot \text{m} \cdot \text{s}^2$ .



- C. (10 points) the system shown below experiences the base excitation  $y(t) = Y_0 \sin \omega t$  where  $Y_0 = 0.5 \text{ mm}$  and  $\omega = 200 \text{ rad/s}$ . Using the undamped approximation for light damping, compute the steady state amplitudes of the motions of the two masses (i.e. compute the steady state amplitudes of  $x_1$  and  $x_2$ ). The masses and spring constants are  $k_1 = k_3 = 4 \text{ N}/\text{m}$ ,  $k_2 = 2 \text{ N}/\text{m}$ ,  $m_1 = 0.01 \text{ kg}$ ,  $m_2 = 0.02 \text{ kg}$ .



# Problem 4

(A.)



$$\begin{aligned} \sum M &= -J_0 \ddot{\theta} - 3(k\theta r)r + k(x - \theta r)r = 0 \\ \sum F &= -m\ddot{x} - c\dot{x} - k(x - \theta r) = 0 \end{aligned}$$

$$\begin{bmatrix} J_0 & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} 4kr^2 & -kr \\ -kr & k \end{bmatrix} \begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(B.)

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det \begin{bmatrix} -\omega^2 J_1 + (k_{t1} + k_{t2}) & -k_{t2} \\ -k_{t2} & -\omega^2 J_2 + k_{t2} \end{bmatrix} = 0$$

$$\det \begin{bmatrix} -2\omega^2 + 300 & -200 \\ -200 & -4\omega^2 + 200 \end{bmatrix} = 0$$

PROB 4 CONT - (B.) CONT -

$$(-2\omega^2 + 300)(-4\omega^2 + 200) - (200)^2 = 0$$

$$8\omega^4 - 1600\omega^2 + 20000 = 0$$

$$\omega^4 = \frac{1600}{16} \pm \frac{1}{16} \sqrt{(-1600)^2 - 4(8)(20000)}$$

$$= 100 \pm 86.6$$

$$\omega_1^2 = 186.6 \text{ rad/s}^2 \quad \omega_2^2 = 13.4 \text{ rad/s}^2$$

$$\omega_1 = 13.66 \text{ rad/s} \quad \omega_2 = 3.66 \text{ rad/s}$$

mode 1

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}}$$

$$\omega_1^2 = 186.6 \frac{\text{rad}}{\text{s}^2}$$

$$\begin{bmatrix} -2\omega_1^2 + 300 & -200 \\ -200 & -4\omega_1^2 + 200 \end{bmatrix} \begin{Bmatrix} \Theta_1^{(1)} \\ r_1 \Theta_1^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$r_1 = \frac{-2(186.6) + 300}{200} = \boxed{-0.366}$$

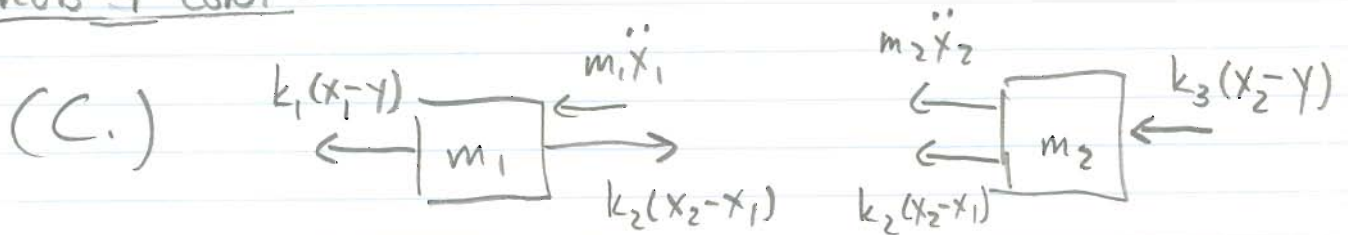
mode 2

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}}$$

$$\omega_2^2 = 13.4 \text{ rad/s}^2$$

$$r_2 = \frac{-2(13.4) + 300}{200} = \boxed{1.366}$$

Prob 4 cont -



$$\sum_1 F = -m_1 \ddot{x}_1 - k_1(x_1 - y) + k_2(x_2 - x_1) = 0$$

$$\sum_2 F = -m_2 \ddot{x}_2 - k_2(x_2 - x_1) - k_3(x_2 - y) = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} k_1 \\ k_3 \end{Bmatrix} y(t)$$

$$\begin{aligned} k_1 = k_3 = 4 \text{ N/m} & \quad m_1 = 0.01 \text{ kg} & y_0 = 0.005 \text{ m} & = \begin{Bmatrix} k_1 y_0 \\ k_3 y_0 \end{Bmatrix} \sin \omega t \\ k_2 = 2 \text{ N/m} & \quad m_2 = 0.02 \text{ kg} & & \end{aligned}$$

$$\begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0.02 \\ 0.02 \end{Bmatrix} \sin \omega t$$

assume  $x_1 = X_1 \sin \omega t$ ,  $x_2 = X_2 \sin \omega t$

$$\begin{bmatrix} -0.01\omega^2 + 6 & -2 \\ -2 & -0.02\omega^2 + 6 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0.02 \\ 0.02 \end{Bmatrix}$$

$$\omega = 200 \text{ rad/s}$$

Prob 4 cont - 
$$\begin{bmatrix} -394 & -2 \\ -2 & -794 \end{bmatrix} \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{Bmatrix} = \begin{Bmatrix} 0.02 \\ 0.02 \end{Bmatrix}$$

$$\bar{x}_1 = \frac{0.02(-794) - (-2)(0.02)}{-394(-794) - 4} = \boxed{-5.06 \times 10^{-5} \text{ m}}$$

$$\bar{x}_2 = \frac{-394(0.02) - 0.02(-2)}{312832} = \boxed{-2.506 \times 10^{-5} \text{ m}}$$

$$|\bar{x}_1| = 5.06 \times 10^{-5} \text{ m} ; |\bar{x}_2| = 2.506 \times 10^{-5} \text{ m}$$

## AE/ME 455 Exam I Formula Sheet

### Undamped Free Vibrations

$$\ddot{x} + \omega_n^2 x = 0$$

$$x(t) = C \cos(\omega_n t - \phi)$$

$$x(t) = C \sin(\omega_n t + \phi_o)$$

$$x(t) = A \cos \omega_n t + B \sin \omega_n t$$

$$C = \sqrt{A^2 + B^2}; \quad \phi = \tan^{-1}\left(\frac{B}{A}\right); \quad \phi_o = \tan^{-1}\left(\frac{A}{B}\right)$$

### Damped Free Vibrations

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$$

1.  $\zeta < 1$  Underdamped Case:

$$x(t) = X e^{-\zeta\omega_n t} \cos(\omega_d t - \phi)$$

$$x(t) = X e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_o)$$

$$x(t) = e^{-\zeta\omega_n t} [A_1 \cos \omega_d t + B_1 \sin \omega_d t]$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$X = \sqrt{A_1^2 + B_1^2}; \quad \phi = \tan^{-1}\left(\frac{B_1}{A_1}\right); \quad \phi_o = \tan^{-1}\left(\frac{A_1}{B_1}\right)$$

2.  $\zeta = 1$  Critically Damped Case:

$$x(t) = (C_1 + C_2 t) e^{-\omega_n t}$$

3.  $\zeta > 1$  Overdamped Case:

$$x(t) = (C_1 e^{(\sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{-(\sqrt{\zeta^2 - 1})\omega_n t}) e^{-\zeta\omega_n t}$$

### Other Formulae:

$$\text{period} = \tau = \frac{1}{f} = \frac{2\pi}{\omega}$$

$f$  = frequency in Hz

$\omega$  = frequency in rad/s

AE/ME 455 Formula Page for Exam II

Simple harmonic function

$$z(t) = a \cos \omega t + b \sin \omega t = C \cos(\omega t - \psi) = C \sin(\omega t + \psi_o)$$

$$C = \sqrt{a^2 + b^2}; \quad \psi = \tan^{-1}\left(\frac{b}{a}\right); \quad \psi_o = \tan^{-1}\left(\frac{a}{b}\right)$$

Harmonically excited single degree of freedom system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \omega_n^2\delta \sin \omega t$$

$$x_{ss}(t) = X_{ss} \sin(\omega t - \phi)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \omega_n^2\delta \cos \omega t$$

$$x_{ss}(t) = X_{ss} \cos(\omega t - \phi)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \omega_n^2\delta \cos(\omega t - \beta)$$

$$x_{ss}(t) = X_{ss} \cos(\omega t - \beta - \phi)$$

$$X_{ss} = \frac{\delta}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}; \quad \phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right); \quad r = \frac{\omega}{\omega_n}$$

### Harmonically excited single degree of freedom system - base excitation

Base excitation has form  $\sin \omega t$  :

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \omega_n^2Y[2\zeta r \cos \omega t + \sin \omega t] = \omega_n^2\bar{Y} \cos(\omega t - \alpha)$$

$$\bar{Y} = Y\sqrt{1 + (2\zeta r)^2}; \quad \alpha = \tan^{-1}\left(\frac{1}{2\zeta r}\right)$$

$$x_{ss}(t) = Q_{ss} \cos(\omega t - \alpha - \phi)$$

$$Q_{ss} = \frac{\bar{Y}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}; \quad \phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right); \quad r = \frac{\omega}{\omega_n}$$

Base excitation has form  $\cos \omega t$  :

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \omega_n^2Y[\cos \omega t - 2\zeta r \sin \omega t] = \omega_n^2\bar{Y} \sin(\omega t - \alpha)$$

$$\bar{Y} = Y\sqrt{1 + (2\zeta r)^2}; \quad \alpha = \tan^{-1}\left(\frac{1}{2\zeta r}\right)$$

$$x_{ss}(t) = Q_{ss} \sin(\omega t - \alpha - \phi)$$

$$Q_{ss} = \frac{\bar{Y}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}; \quad \phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right); \quad r = \frac{\omega}{\omega_n}$$

### Periodic excitation represented by Fourier series

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \omega_n^2\delta\left[\frac{a_o}{2} + \sum_{j=1}^{\infty} a_j \cos(j\omega t) + \sum_{j=1}^{\infty} b_j \sin(j\omega t)\right]$$

$$x_{ss}(t) = \delta\left[\frac{a_o}{2} + \sum_{j=1}^{\infty} \frac{a_j}{\sqrt{(1-j^2r^2)^2 + (2\zeta jr)^2}} \cos(j\omega t - \phi_j) + \sum_{j=1}^{\infty} \frac{b_j}{\sqrt{(1-j^2r^2)^2 + (2\zeta jr)^2}} \sin(j\omega t - \phi_j)\right]$$

$$r = \frac{\omega}{\omega_n}; \quad jr = \frac{j\omega}{\omega_n}; \quad \phi_j = \tan^{-1}\left(\frac{2\zeta jr}{1-j^2r^2}\right)$$

### Fourier Series

$F(t)$  periodic with period =  $\tau$ ; fundamental frequency =  $\omega = \frac{2\pi}{\tau}$

Fourier series for  $F(t)$  is:

$$F(t) = \frac{a_o}{2} + \sum_{j=1}^{\infty} a_j \cos(j\omega t) + \sum_{j=1}^{\infty} b_j \sin(j\omega t)$$

$$a_o = \frac{\omega}{\pi} \int_0^{2\pi/\omega} F(t) dt; \quad a_j = \frac{\omega}{\pi} \int_0^{2\pi/\omega} F(t) \cos(j\omega t) dt; \quad b_j = \frac{\omega}{\pi} \int_0^{2\pi/\omega} F(t) \sin(j\omega t) dt$$